The unitarity method using a canonical basis approach

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ABSTRACT: Various implementations of the Unitarity method have been developed to compute one-loop amplitudes in gauge theories. In this paper we present an implementation which uses canonical forms to generate the rational coefficients of the basis integral functions. As an example, we present the results for the $\mathcal{N} = 1$ contribution to seven gluon scattering in closed, rational, analytic form.

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Contents

1 Introduction

One-loop computations are an essential ingredient in providing robust next-to-leading order predictions for QCD events at colliders such as the LHC [1]. A general one-loop amplitude
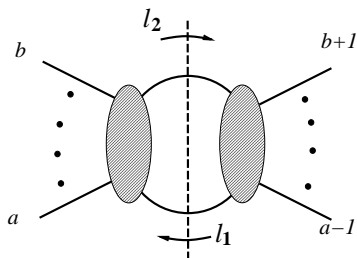
Figure 1. A two-particle cut of a one-loop amplitude

for massless particles can be expressed, after an appropriate Passarino-Veltman reduction [2], in terms of n-point scalar integral functions I_n^i with rational coefficients, a_i, b_j, c_k,

$$\mathcal{A}_n^{1-\text{loop}} = \sum_{i \in \mathcal{C}} a_i \, I_4^i + \sum_{j \in \mathcal{D}} b_j \, I_3^j + \sum_{k \in \mathcal{E}} c_k \, I_2^k + R_n + O(\epsilon) \,. \tag{1.1}$$

The functions I_n^i contain all of the logarithms and dilogarithms in the amplitude and R_n is the remaining rational part. The summations are over all possible integral functions. The Unitarity technique [3, 4] determines the rational coefficients of these functions from the information contained in the two-particle cuts shown in figure 1, using physical on-shell amplitudes as inputs.

The cut,

$$C_{a,\ldots,b} \equiv \frac{i}{2} \int d\text{LIPS}\Big[\mathcal{A}^{\text{tree}}(-\ell_1, a, a+1, \ldots, b, \ell_2) \times \mathcal{A}^{\text{tree}}(-\ell_2, b+1, b+2, \ldots, a-1, \ell_1)\Big] \,, \tag{1.2}$$

where $\int d\text{LIPS}$ denotes integration over the on-shell phase space of the ℓ_i is equal to the leading discontinuity in the integral functions [5] of eq. (1.1),

$$C_{a,\ldots,b} = \left(\sum_{i \in \mathcal{C}'} a_i \, I_4^i + \sum_{j \in \mathcal{D}'} b_j \, I_3^j + \sum_{k \in \mathcal{E}'} c_k \, I_2^k\right)\Big|_{\text{Disc}} \,, \tag{1.3}$$

where $\mathcal{C}'$, $\mathcal{D}'$ and $\mathcal{E}'$ are the restricted sets of functions with a cut in this particular channel. In particular, $\mathcal{E}'$ consists of a single term.

The original implementation of the Unitarity method did not evaluate the cut directly but instead manipulated the product of tree amplitudes, using $\ell_1^2 = \ell_2^2 = 0$, to rewrite it in the form,

$$\int d\text{LIPS}\mathcal{A}^{\text{tree}}(-\ell_1, a, a+1, \ldots, b, \ell_2) \times \mathcal{A}^{\text{tree}}(-\ell_2, b+1, b+2, \ldots, a-1, \ell_1)$$
$$= \int d\text{LIPS}\left(\sum a_i \frac{1}{(\ell_1 - K_1^i)^2(\ell_1 - K_2^i)^2} + \sum b_j \frac{1}{(\ell_1 - K_3^j)^2} + c_k\right), \tag{1.4}$$

and identified the coefficients in the above with the integral coefficients of eq. (1.3). A feature of the original implementation is that the representation (1.4) is not unique but one must simultaneously solve the full set of cut equations.

In principle, the cut momenta ℓ_i should match the momenta of the integral functions I_n, i.e. they should be in $4-2\epsilon$ dimensions, however it was shown that for many amplitudes it is sufficient to use four dimensional tree amplitudes. Using four dimensional amplitudes allows us to evaluate the integral coefficients but not the rational terms R_n. For supersymmetric amplitudes, or the supersymmetric components of QCD amplitudes, $R_n = 0$ and we term these amplitudes "cut-constructible" [3].

In recent years considerable advances have been made in systematising the process of extracting the coefficients of the basis integral functions. Progress has been made both via the two-particle cuts above and using generalisations of unitarity [6] where, for example, triple [7–10] and quadruple cuts [11] are utilised to identify the triangle and box coefficients. Triple and quadruple cuts are useful in that they isolate contributions from smaller sets of integral functions. For example, a quadruple cut isolates a single box coefficient. Since the cut inserts four δ-functions into the covariant integral the coefficient of this box function is given by the algebraic product of four tree amplitudes [11] . $\mathcal{N} = 4$ one-loop amplitudes consist solely of scalar box functions and so quadruple cuts are sufficient to completely compute them [12–15]: a property shared by $\mathcal{N} = 8$ supergravity amplitudes [16].

The triple cut with three δ-functions is effectively a one-parameter integral which can be evaluated via complex methods [8, 9]. One may also consider one-particle cuts [17]. Generalised Unitarity has been used beyond one-loop at two loops [18, 19] and beyond [20].

Our strategy is to use all possible cuts and first evaluate the box coefficients from quadruple cuts then the triangle coefficients from triple cuts and finally use the two-particle cuts to determine the bubble coefficients. This is not the only strategy since the two-particle cuts contain enough information to determine the coefficients of the box and triangle contributions as well as the bubble coefficients. The testing ground for many of these techniques has been the computation of the various terms in six gluon one-loop scattering amplitudes [21] both for the supersymmetric contributions [3, 4, 22, 23] and also for the "cut-constructible" parts of the QCD amplitudes [4, 24, 25]. For full QCD amplitudes the rational terms must also be calculated. Unitarity can be used to determine these, however this requires the use of tree amplitudes defined in $D = 4 - 2\epsilon$ dimensions [26]. Alternatively, the rational pieces can be obtained using on-shell recursion [27], a method akin to that for tree amplitudes [28]. This has a numerical implementation together with Generalised Unitarity [29]. One may also use specialised Feynman diagram techniques which focus on the rational terms [30–32].

The approach we adopt recognises that there are a limited number of distinct structures that appear in the cut integrals. These may be evaluated in a number of ways: conventional covariant integration, fermionic integration [23, 25], direct extraction [8] or integrand level reduction [33]. By determining the contribution of each structure to the relevant coefficients we construct a canonical basis which, once constructed, can be used for any amplitude. While this approach reproduces results from other methods, the decomposition into canonical forms is carried out directly on the four dimensional tree amplitudes without re-parametrising the $dLIPS$ integration. It produces compact, explicitly rational expressions for the integral coefficients. We illustrate this process by presenting the $\mathcal{N} = 1$

contribution to one-loop seven gluon scattering in closed analytic form.[1]

2   Organisation of the amplitudes

The organisation of loop amplitudes into physical sub-amplitudes is an important step toward computing these amplitudes: although eventually all the pieces must be reassembled. For one-loop amplitudes with adjoint particles, one may perform a *colour decomposition* similar to the tree-level decomposition [34]. This one-loop decomposition is [35],

$$\mathcal{A}_n^{1-\text{loop}} = ig^n \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n / S_{n;c}} \text{Gr}_{n;c}(\sigma) \, A_{n;c}(\sigma) \,, \tag{2.1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.[2] The leading colour-structure factor,

$$\text{Gr}_{n;1}(1) \;=\; N_c \; \text{Tr}\,(T^{a_1} \cdots T^{a_n}) \,, \tag{2.2}$$

is just N_c times the tree colour factor and the sub-leading colour structures ($c > 1$) are given by,

$$\text{Gr}_{n;c}(1) \;=\; \text{Tr}\,(T^{a_1} \cdots T^{a_{c-1}}) \; \text{Tr}\,(T^{a_c} \cdots T^{a_n}) \,. \tag{2.3}$$

S_n is the set of all permutations of n objects and $S_{n;c}$ is the subset leaving $\text{Gr}_{n;c}$ invariant [35]. The contributions from fundamental representation quarks circulating in the loop can be obtained from the same partial amplitudes, except that the sum only runs over the $A_{n;1}$ and the overall factor of N_c in $\text{Gr}_{n;1}$ is dropped. For one-loop amplitudes of gluons the $A_{n;c}$, $c > 1$ can be obtained from the $A_{n;1}$ by summing over permutations [3, 35]. Hence it is sufficient to compute $A_{n;1}$ in what follows and, for clarity, we refer to these as A_n. The partial amplitudes A_n have cyclic symmetry rather than full crossing symmetry.

We choose to use a *supersymmetric decomposition*. Instead of calculating the one-loop contributions from massless gluons, $A_n^{[1]}$, or quarks, $A_n^{[1/2]}$, circulating in the loop, it is considerably more convenient to calculate the contributions from the full $\mathcal{N} = 4$ multiplet, a $\mathcal{N} = 1$ chiral multiplet and a complex scalar circulating in the loop. In terms of these,

$$\begin{aligned}
A_n^{[1]} &= A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1\ \text{chiral}} + A_n^{[0]} \,, \\
A_n^{[1/2]} &= A_n^{\mathcal{N}=1\ \text{chiral}} - A_n^{[0]} \,.
\end{aligned} \tag{2.4}$$

The amplitudes are also organised according to the helicities of the outgoing gluons which may be $\pm$. We use polarisation tensors formed from Weyl spinors [36],

$$\epsilon_\mu^+(k;q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2} \langle q\,k \rangle} \,, \; \epsilon_\mu^-(k;q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2} [k\,q]} \,, \tag{2.5}$$

where k is the gluon momentum and q is an arbitrary null 'reference momentum' which drops out of the final gauge-invariant amplitudes. The plus and minus labels on the polarization vectors refer to the gluon helicities and we use the notation $\langle ij \rangle \equiv \langle k_i^- | k_j^+ \rangle$, $[ij] \equiv$

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[1]These are available in `Mathematica` format at http://pyweb.swan.ac.uk/~dunbar/sevengluon.html

[2]We have inserted a factor of i in this definition to avoid universal factors of i appearing in our explicit formulae.

$\langle k_i^+ | k_j^- \rangle$. In twistor-inspired studies of gauge theory amplitudes [37] the two component Weyl spinors are often expressed as,

$$\lambda_a \equiv |k^+\rangle\,, \quad \bar{\lambda}_{\dot{a}} \equiv |k^-\rangle. \tag{2.6}$$

Helicity amplitudes are related to those with all legs of opposite helicity by conjugation: $\langle a\,b \rangle \leftrightarrow [b\,a]$. Using spinor helicity leads to amplitudes which are functions of the spinor variables $\langle a\,b \rangle$ and $[a\,b]$ and combinations such as $\langle k_i^+ | \not{p} | k_j^+ \rangle = [k_i|p|k_j\rangle \equiv [k_i\,p]\langle p\,k_j \rangle$. It is useful to define combinations of spinor products,

$$[a|P_{b\cdots f}|m\rangle \equiv [a\,b]\langle b\,m \rangle + \cdots + [a\,f]\langle f\,m \rangle\,. \tag{2.7}$$

In this article we complete the computation of the $\mathcal{N}=1$ contributions to seven gluon scattering. Up to conjugation and relabeling, there are nine independent helicity configurations for the colour ordered amplitudes. The amplitudes $A_7(1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+)$ and $A_7(1^-, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+)$ vanish to all orders in perturbation theory within any supersymmetry theory so,

$$A_7^{\mathcal{N}=1}(1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+) = A_7^{\mathcal{N}=1}(1^-, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+) = 0\,. \tag{2.8}$$

Amplitudes with exactly two negative helicities are referred to as MHV ("maximally helicity violating") amplitudes and those with three negative helicities as NMHV ("next to MHV") amplitudes. There are three independent MHV and four independent NMHV helicity configurations for the seven gluon amplitude. The seven-point MHV amplitudes [4] and the NMHV amplitude with the three negative helicity legs adjacent $A_7^{\mathcal{N}=1}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ [7] are specific cases of known all-n expressions. We present explicit forms for the remaining three NMHV partial amplitudes which will be made available at http://pyweb.swan.ac.uk/~dunbar/sevengluon.html in `Mathematica` format.

3 Canonical basis for bubble coefficients from two-particle cuts

Consider the general decomposition of the product of tree amplitudes appearing in a two-particle cut in terms of canonical forms $\mathcal{F}_i$,

$$A^{\text{tree}}(-\ell_1, \ldots, \ell_2) \times A^{\text{tree}}(-\ell_2, \ldots, \ell_1) = \sum c_i \mathcal{F}_i(\ell_j), \tag{3.1}$$

where the c_i are coefficients independent of ℓ_j. The forms $\mathcal{F}_i$ must have zero spinor weight in the ℓ_j, i.e. they must be invariant under $|\ell_j\rangle \longrightarrow e^{i\phi_j}|\ell_j\rangle, |\ell_j] \longrightarrow e^{-i\phi_j}|\ell_j]$. Each two-particle cut receives contributions from box functions, triangle functions and a bubble. The coefficients of the box functions are most simply determined from quadruple cuts and the triangle coefficients from triple cuts, so we only use the two-particle cut to determine the bubble coefficients. Consequently, we must determine the contributions to the bubble coefficients from the various canonical forms, $\mathcal{F}_i$.

We examine the $\mathcal{F}_i$ according to their overall order in the cut momenta ℓ_i. In general, terms of order ℓ^N with $N < 0$ only give contributions to box and triangle functions so we are interested in terms of order ℓ^N with $N \geq 0$. For a generic QCD amplitude the contributions are typically of order ℓ^2. In the $\mathcal{N}=1$ contributions to Yang-Mills amplitudes cancellations generically reduce this to order ℓ^0.

3.1 Order ℓ^0 terms

These are the canonical forms which are required to obtain the $\mathcal{N} = 1$ contributions to Yang-Mills amplitudes.

Functionally we start with the simplest non-trivial case,

$$\mathcal{H}_1(A; B; \ell_1) \equiv \frac{\langle \ell_1 B \rangle}{\langle \ell_1 A \rangle} = -\frac{[A|\ell_1|B\rangle}{(\ell_1 - k_A)^2}, \tag{3.2}$$

where k_A is taken to be real. There are many ways to evaluate the contribution to the bubble coefficient from this simple form. We chose to manipulate this as if it were a covariant integral. This means effectively replacing,

$$\int dLIPS \longrightarrow \int \frac{d^D \ell}{\ell_1^2 \ell_2^2}, \tag{3.3}$$

then evaluating the covariant integral only keeping the coefficient of $-\ln(-P^2)$ in the result. The integral is then a linear triangle integral with massless leg k_A as shown,
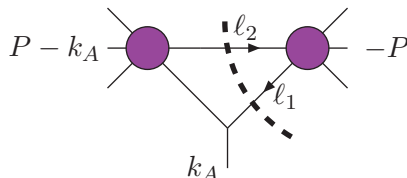
which can be evaluated by shifting the momenta using Feynman parameters,

$$\ell_1^\mu \longrightarrow \ell_1^\mu - k_A^\mu a_2 + P^\mu a_3, \tag{3.4}$$

where $P \equiv k_a + k_{a+1} + \cdots k_b$ is the total momentum across the cut and a_3 refers to the Feynman parameter of the ℓ_2 propagator.

The a_2 term drops out of the integral and we use,

$$I_3[a_3] = \frac{\ln(-P^2) - \ln(-(P - k_A)^2)}{2k_A \cdot P}. \tag{3.5}$$

The coefficient of $-\ln(-P^2)$ is the bubble coefficient and thus $\mathcal{H}_1$ evaluates to H_1,

$$H_1[A; B; P] = \frac{[A|P|B\rangle}{2k_A \cdot P} = \frac{[A|P|B\rangle}{[A|P|A\rangle}. \tag{3.6}$$

Note we assume in the above that k_A is real. If A denoted a complex combination of momenta then $|A]$ should be replaced by $|A\rangle^*$ in the canonical form. We also have the conjugate result,

$$\bar{\mathcal{H}}_1(C; D; \ell_1) = \frac{[D\,\ell_1]}{[C\,\ell_1]} \implies \bar{H}_1[C; D; P] = \frac{[D|P|C\rangle}{[C|P|C\rangle}. \tag{3.7}$$

These forms satisfy,

$$H_1[\,P|A]; B; P] = \frac{1}{P^2} \bar{H}_1[A; P|B\rangle; P]. \tag{3.8}$$

We have chosen to determine the contribution from this form using covariant integrals, of course one obtains the same result by applying fermionic integration [23] or direct evaluation [8]. Our aim is to use this simple result for H_1 as the starting point for generating many further terms. First consider $\mathcal{H}$ to be a holomorphic function of one of the ℓ_i, i.e. $\mathcal{H} = \mathcal{H}(|\ell\rangle)$, then we can define,

$$\mathcal{H}_n(A_i; B_j; \ell) = \frac{\prod_{j=1}^n \langle B_j\, \ell\rangle}{\prod_{i=1}^n \langle A_i\, \ell\rangle}, \quad \langle A_i\, A_j\rangle \neq 0. \tag{3.9}$$

By spinor weight the number of A_i is the number of B_j. The cases where multiple poles appear are treated separately: such terms cancel in *amplitudes* by the factorisation theorems.

By expressing this as a partial fraction we can split $\mathcal{H}_n$ algebraically into a sum of $\mathcal{H}_1$ terms,

$$\mathcal{H}_n(A_i; B_i; \ell) = \sum_i c_i \frac{\langle B_1\, \ell\rangle}{\langle A_i\, \ell\rangle} = \sum_i c_i \mathcal{H}_1(A_i; B_1; \ell), \tag{3.10}$$

where the coefficients c_i are given by,

$$c_i = \frac{\prod_{j=2}^n \langle B_j\, A_i\rangle}{\prod_{j\neq i} \langle A_j\, A_i\rangle}. \tag{3.11}$$

The bubble coefficient generated by $\mathcal{H}_n$ is thus,

$$H_n[A_i; B_i; P] = \sum_i \frac{\prod_{j=2}^n \langle B_j\, A_i\rangle}{\prod_{j\neq i} \langle A_j\, A_i\rangle} \frac{\langle B_1|P|A_i]}{\langle A_i|P|A_i]}. \tag{3.12}$$

The same formula applies whether we have a holomorphic expression in ℓ_1 or ℓ_2. When we have a mixed expression,

$$\frac{\prod_{i=1}^n \langle B_i\, \ell_1\rangle \prod_{j=1}^m \langle C_j\, \ell_2\rangle}{\prod_{i=1}^n \langle A_i\, \ell_1\rangle \prod_{j=1}^m \langle D_j\, \ell_2\rangle} \frac{\prod_{k=1}^p [E_k\, \ell_1] \prod_{l=1}^q [G_l\, \ell_2]}{\prod_{k=1}^p [F_k\, \ell_1] \prod_{l=1}^q [H_l\, \ell_2]}, \tag{3.13}$$

we first rewrite it as a holomorphic expression in ℓ_1 and ℓ_2,

$$\frac{\prod_{i=1}^n \langle B_i\, \ell_1\rangle \prod_{j=1}^m \langle C_j\, \ell_2\rangle}{\prod_{i=1}^n \langle A_i\, \ell_1\rangle \prod_{j=1}^m \langle D_j\, \ell_2\rangle} \frac{\prod_{k=1}^p [E_k|P|\ell_2\rangle \prod_{l=1}^q [G_l|P|\ell_1\rangle}{\prod_{k=1}^p [F_k|P|\ell_2\rangle \prod_{l=1}^q [H_l|P|\ell_1\rangle}, \tag{3.14}$$

then use the identity,

$$\frac{\langle a\, \ell_2\rangle}{\langle b\, \ell_2\rangle} = \frac{\langle a\, \ell_1\rangle}{\langle b\, \ell_1\rangle} - \frac{P^2 \langle a\, b\rangle}{\langle b\, \ell_1\rangle [\ell_1|P|b\rangle}, \tag{3.15}$$

to replace it by,

$$\frac{\prod_{i=1}^n \langle B_i\, \ell_1\rangle \prod_{j=1}^m \langle C_j\, \ell_1\rangle}{\prod_{i=1}^n \langle A_i\, \ell_1\rangle \prod_{j=1}^m \langle D_j\, \ell_1\rangle} \frac{\prod_{k=1}^p [E_k|P|\ell_1\rangle \prod_{l=1}^q [G_l|P|\ell_1\rangle}{\prod_{k=1}^p [F_k|P|\ell_1\rangle \prod_{l=1}^q [H_l|P|\ell_1\rangle} + \text{terms of order } \ell^{-2}. \tag{3.16}$$

Only the leading term contributes to the bubble coefficient and it gives an overall contribution of $H_{n+m+p+q}[A_i, D_j, P|F_k], P|H_l]; B_i, C_j, P|E_k], P|G_l]; P]$, provided the $A_i, D_j, P|F_k]$ and $P|H_l]$ are all distinct.

We can have terms with $D = A$. In this case we decompose into terms of the form,

$$\mathcal{H}_2^x(A, A; B_1, B_2) = \frac{\langle B_1 \ell_1 \rangle \langle B_2 \ell_2 \rangle}{\langle A \ell_1 \rangle \langle A \ell_2 \rangle}. \tag{3.17}$$

This special case canonical form gives,

$$H_2^x[A, A; B_1, B_2; P] = \frac{[A|P|B_1\rangle[A|P|B_2\rangle}{[A|P|A\rangle^2}. \tag{3.18}$$

There is a second class of functions arising from terms with *massive* propagators such as,

$$\mathcal{G}_0(B; D; Q; \ell_1) = \frac{1}{(\ell_1 + Q)^2}[D|\ell_1|B\rangle, \tag{3.19}$$

where $Q^2 \neq 0$. We can relate this to the $\mathcal{H}_1$ form using the identity,

$$\frac{1}{(\ell + Q)^2}[D \ell] = \frac{1}{(\ell + Q)^2}\frac{[D|P(P + Q)Q|\ell\rangle}{\langle \ell|PQ|\ell\rangle} - \frac{[D|P|\ell\rangle}{\langle \ell|PQ|\ell\rangle}, \tag{3.20}$$

leading to,

$$\frac{1}{(\ell_1 + Q)^2}[D|\ell_1|B\rangle = -\frac{[D|P|\ell_1\rangle\langle B \ell_1\rangle}{\langle \ell_1|PQ|\ell_1\rangle} + \text{sub-leading.} \tag{3.21}$$

We then make the replacement,

$$\langle \ell|PQ|\ell\rangle \sim \langle \ell|\hat{P}\hat{Q}|\ell\rangle = \langle \ell \hat{P}\rangle[\hat{P}\hat{Q}]\langle \hat{Q} \ell\rangle, \tag{3.22}$$

where $\hat{P}$ and $\hat{Q}$ are the null linear combinations of P and Q,

$$\hat{P}^\mu = \frac{1}{2\sqrt{\Delta_3}}\left(P^2 Q^\mu - \left(P \cdot Q - \frac{\sqrt{\Delta_3}}{2}\right)P^\mu\right), \quad \hat{Q}^\mu = \frac{1}{2\sqrt{\Delta_3}}\left(-P^2 Q^\mu + \left(P \cdot Q + \frac{\sqrt{\Delta_3}}{2}\right)P^\mu\right), \tag{3.23}$$

with $\Delta_3 = \Delta_3(P, Q) = 4(P \cdot Q)^2 - 4P^2 Q^2$. Δ_3 is the Gram determinant of the three-mass triangle integral having legs of momenta P, Q and $-P - Q$. The leading term in eq. (3.21) then has precisely the form of an $\mathcal{H}_2$ function. Splitting this into a pair of $\mathcal{H}_1$ functions gives two terms that are not individually rational because $\hat{P}$ and $\hat{Q}$ contain factors of $\sqrt{\Delta_3}$. The pair of terms are however the irrational conjugates of each other, so the sum is rational. To have canonical forms which yield explicitly rational coefficients we choose to evaluate $\mathcal{G}_0$ as a separate canonical form which is manifestly rational:

$$G_0[B; D; Q; P] = \frac{[D|P(QP - PQ)|B\rangle}{\Delta_3} = \frac{[D|P[Q, P]|B\rangle}{\Delta_3}. \tag{3.24}$$

We commonly find the form,

$$\mathcal{G}_1(A; B_0, B_1; D; Q; \ell_1) = \frac{1}{(\ell_1 + Q)^2}\frac{[D|\ell_1|B_0\rangle\langle \ell_1 B_1\rangle}{\langle \ell_1 A\rangle}. \tag{3.25}$$

For $\langle A \hat{P}\rangle, \langle A \hat{Q}\rangle \neq 0$, this can be decomposed as an $\mathcal{H}_3$ function but at the cost of introducing irrational coefficients. Once again, combining these terms generates a manifestly

rational form. For $\langle A\,\hat{P}\rangle, \langle A\,\hat{Q}\rangle \neq 0$,

$$
\begin{aligned}
G_1[A; B_0, B_1; D; Q; P] = &- \frac{[D|P[P,Q]|A\rangle \langle B_1|[P,Q]|B_0\rangle}{2\Delta_3 \langle A|PQ|A\rangle} \\
&+ \frac{[D|P|A\rangle (\langle B_0\,A\rangle \langle B_1|P|A] + \langle B_1\,A\rangle \langle B_0|P|A])}{2\langle A|PQ|A\rangle \langle A|P|A]} .
\end{aligned}
\tag{3.26}
$$

We can extend this form to determine the bubble contributions arising from,

$$
\mathcal{G}_n(A_i; B_0, B_i; D; Q; \ell_1) = \frac{1}{(\ell_1 + Q)^2} \frac{[D|\ell_1|B_0\rangle \prod_{i=1}^n \langle \ell_1\,B_i\rangle}{\prod_{i=1}^n \langle \ell_1\,A_i\rangle},
\tag{3.27}
$$

by splitting it into a sum of $\mathcal{G}_1$ terms, just as we split $\mathcal{H}_n$ into a sum of $\mathcal{H}_1$ terms,

$$
\mathcal{G}_n(A_i; B_0, B_i; D; Q; \ell_1) = \sum_i c_i\, \mathcal{G}_1(A_i; B_0, B_n; D; Q; \ell_1),
\tag{3.28}
$$

with,

$$
c_i = \frac{\prod_{j<n} \langle A_i\,B_j\rangle}{\prod_{j\neq i} \langle A_i\,A_j\rangle}.
\tag{3.29}
$$

The H_n and G_n functions are sufficient to evaluate the bubble coefficients of the $\mathcal{N} = 1$ contributions to Yang-Mills amplitudes. We will show this by example in the following sections where we explicitly evaluate the seven-point $\mathcal{N} = 1$ one-loop contributions.

In general we may also have terms with multiple massive propagators,

$$
\frac{f(\ell)}{(\ell + Q_1)^2 (\ell + Q_2)^2 \dots}.
\tag{3.30}
$$

Using the constraint $(\ell - P)^2 = 0$, we have,

$$
(\ell + Q_1)^2 = [\ell|Q_1 + \frac{Q_1^2}{P^2} P|\ell\rangle \equiv [\ell|\mathcal{Q}_1|\ell\rangle,
\tag{3.31}
$$

where $\mathcal{Q}_1$ is a non-null linear combination of Q_1 and P. This allows any multiple propagator terms to be written as,

$$
\frac{f(\ell)}{[\ell|\mathcal{Q}_1|\ell\rangle [\ell|\mathcal{Q}_2|\ell\rangle \dots}.
\tag{3.32}
$$

Partial fractioning on the $\bar{\lambda}(\ell)$'s then gives a sum of terms of the form,

$$
\frac{g(\ell)}{[\ell|\mathcal{Q}_1|\ell\rangle \langle \ell|\mathcal{Q}_1\mathcal{Q}_2|\ell\rangle \dots} \sim \frac{g(\ell)}{[\ell|\mathcal{Q}_1|\ell\rangle \langle \ell\hat{\mathcal{Q}}_1\rangle [\hat{\mathcal{Q}}_1\hat{\mathcal{Q}}_2]\langle \hat{\mathcal{Q}}_2\ell\rangle \dots},
\tag{3.33}
$$

which can be further split into a sum of $\mathcal{G}_1$ forms at the expense of introducing irrational factors in $\sqrt{\Delta_3(\mathcal{Q}_i, \mathcal{Q}_j)}$. As with the $\sqrt{\Delta_3(P, Q)}$ terms, these all arise in irrational conjugate pairs and the sum is rational.

3.2 Terms of order ℓ^1 and ℓ^2

For the scalar contributions to Yang-Mills amplitudes we need forms of order ℓ^1 and ℓ^2. In general we will needs forms denoted H_n^r for contributions of order ℓ^r where the denominator has n factors of $\langle A_i\,\ell \rangle$.

The higher order $\mathcal{H}$ and H forms are:

$$\mathcal{H}_0^1 \equiv [D|\ell_1|B\rangle \to H_0^1[B;D;P] = \frac{1}{2}[D|P|B\rangle,$$

$$\mathcal{H}_1^1 \equiv \frac{[D|\ell_1|B_1\rangle\langle \ell_1\,B_2\rangle}{\langle \ell_1\,A\rangle} \to$$

$$H_1^1[A;B_1,B_2;D;P] = \frac{P^2}{4[A|P|A\rangle^2}\left([D|A|B_1\rangle[A|P|B_2\rangle + [D|A|B_2\rangle[A|P|B_1\rangle\right)$$

$$+ \frac{1}{4[A|P|A\rangle}\left([D|P|B_1\rangle[A|P|B_2\rangle + [D|P|B_2\rangle[A|P|B_1\rangle\right),$$

$$\mathcal{H}_0^2 \equiv [D_1|\ell_1|B_1\rangle[D_2|\ell_1|B_2\rangle \to$$

$$H_0^2[B_1,B_2;D_1,D_2;P] = \frac{1}{3}[D_1|P|B_1\rangle[D_2|P|B_2\rangle + \frac{P^2}{6}[D_1\,D_2]\langle B_1\,B_2\rangle,$$

$$\mathcal{H}_1^2 \equiv \frac{[D_1|\ell_1|B_1\rangle[D_2|\ell_1|B_2\rangle\langle \ell_1\,B_3\rangle}{\langle \ell_1\,A\rangle} \to$$

$$H_1^2[A;B_1,B_2,B_3;D_1,D_2;P] = \frac{(P^2)^2}{18[A|P|A\rangle^3}\left([D_1|A|B_1\rangle[D_2|A|B_2\rangle[A|P|B_3\rangle + \mathcal{P}_6(B_i)\right)$$

$$+ \frac{(P^2)}{36[A|P|A\rangle^2}\left([D_1|P|B_1\rangle[D_2|A|B_2\rangle[A|P|B_3\rangle + \mathcal{P}_{12}(B_i,D_i)\right)$$

$$+ \frac{1}{18[A|P|A\rangle}\left([D_1|P|B_1\rangle[D_2|P|B_2\rangle[A|P|B_3\rangle + \mathcal{P}_6(B_i)\right),$$

$$(3.34)$$

where $\mathcal{P}_6(B_i)$ and $\mathcal{P}_{12}(B_i, D_i)$ represent a total of six and twelve permutations respectively.

The higher order terms involving massive propagators can be reduced to $\mathcal{G}_1^2$ or $\mathcal{G}_1^1$ forms, where,

$$\mathcal{G}_1^n \equiv f^n(\ell)\frac{[D\,\ell]\langle C\,\ell\rangle\langle B\,\ell\rangle}{[\ell|\mathcal{Q}|\ell\rangle\langle A\,\ell\rangle}, \qquad (3.35)$$

and $f^n(\ell)$ is a polynomial of degree n in ℓ. To evaluate these forms we again make use of the identity:

$$\frac{[D\,\ell]}{[\ell|\mathcal{Q}|\ell\rangle} = P^2 \frac{[D|\mathcal{Q}|\ell\rangle}{[\ell|\mathcal{Q}|\ell\rangle\langle \ell|P\mathcal{Q}|\ell\rangle} - \frac{[D|P|\ell\rangle}{\langle \ell|P\mathcal{Q}|\ell\rangle}. \qquad (3.36)$$

This allows us to write,

$$\mathcal{G}_1^n\langle A|P\mathcal{Q}|A\rangle$$

$$= \frac{P^2 f(\ell)\langle B\,\ell\rangle}{[\ell|\mathcal{Q}|\ell\rangle}\left(\frac{[D|\mathcal{Q}|A\rangle\langle C\,A\rangle}{\langle A\,\ell\rangle} + \frac{4}{P^2}\left(\frac{[D|\mathcal{Q}|\hat{P}\rangle\langle C|\hat{P}\hat{\mathcal{Q}}|A\rangle}{\langle \ell\hat{P}\rangle} + \frac{[D|\mathcal{Q}|\hat{\mathcal{Q}}\rangle\langle C|\hat{\mathcal{Q}}\hat{P}|A\rangle}{\langle \ell\hat{\mathcal{Q}}\rangle}\right)\right)$$

$$- f(\ell)\langle B\,\ell\rangle\left(\frac{[D|P|A\rangle\langle C\,A\rangle}{\langle A\,\ell\rangle} + \frac{4}{P^2}\left(\frac{[D|P|\hat{P}\rangle\langle C|\hat{P}\hat{\mathcal{Q}}|A\rangle}{\langle \ell\hat{P}\rangle} + \frac{[D|P|\hat{\mathcal{Q}}\rangle\langle C|\hat{\mathcal{Q}}\hat{P}|A\rangle}{\langle \ell\hat{\mathcal{Q}}\rangle}\right)\right).$$

$$(3.37)$$

This splits the term of interest into pieces with the same overall power count in ℓ but only massless propagators and terms with a reduced power count in ℓ involving the original massive propagator. The former are readily evaluated using our basis of H functions, while the latter rely on G functions. The terms involving the massive propagator along with $\langle \hat{P}\ell\rangle^{-1}$ or $\langle \hat{Q}\ell\rangle^{-1}$ give rise to special cases of the G functions as discussed below. Using this splitting procedure, the order ℓ^1 term, $\mathcal{G}_1^1$, is expressed in terms of a sum involving special cases of the order ℓ^0 $\mathcal{G}_1$ form, namely, $\mathcal{G}_1^{\hat{Q}}$ and $\mathcal{G}_1^{\hat{P}}$, where,

$$\mathcal{G}_1^{\hat{Q}} = \frac{[D\,\ell]\langle C\,\ell\rangle\langle B\,\ell\rangle}{[\ell|\mathcal{Q}|\ell\rangle\langle \hat{\mathcal{Q}}\,\ell\rangle}\;. \tag{3.38}$$

Evaluating this form gives the bubble contribution,

$$G_1^{\hat{Q}} = 2\frac{[D|[\mathcal{Q},P]P|C\rangle[\hat{\mathcal{Q}}|P|B\rangle + \left(P^2[D|\mathcal{Q}|B\rangle - (P.\mathcal{Q}+\sqrt{\Delta_3}/2)[D|P|B\rangle\right)[\hat{\mathcal{Q}}|P|C\rangle}{P^2\Delta_3}. \tag{3.39}$$

$\mathcal{G}_1^{\hat{P}}$ and $G_1^{\hat{P}}$ are obtained by irrational conjugation.

Similarly for the order ℓ^2 terms we require special cases of $\mathcal{G}_1^1$ with $\langle D\,\hat{Q}\rangle = 0$ and $\langle D\,\hat{P}\rangle = 0$. Setting $f(\ell) = [E|\ell|F\rangle$, we have,

$$\mathcal{G}_1^{1:\hat{Q}} = \frac{[E|\ell|F\rangle[D|\ell|C\rangle\langle\ell B\rangle}{[\ell|\mathcal{Q}|\ell\rangle\langle\ell\,\hat{\mathcal{Q}}\rangle}. \tag{3.40}$$

Making the definitions,

$$\begin{aligned}
\beta =& \Big([E|\hat{P}|F\rangle[D|\hat{P}|C\rangle[\hat{\mathcal{Q}}|P|B\rangle - [E|\hat{\mathcal{Q}}|F\rangle[D|\hat{\mathcal{Q}}|C\rangle[\hat{\mathcal{Q}}|P|B\rangle \\
& - [E|\hat{\mathcal{Q}}|B\rangle[D|\hat{\mathcal{Q}}|C\rangle[\hat{\mathcal{Q}}|P|F\rangle - [D|\hat{\mathcal{Q}}|B\rangle[E|\hat{\mathcal{Q}}|F\rangle[\hat{\mathcal{Q}}|P|C\rangle \Big),
\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}
\gamma =& \Big(\Big([E|\hat{P}|F\rangle - [E|\hat{\mathcal{Q}}|F\rangle\Big)\Big([D|\hat{P}|C\rangle - [D|\hat{\mathcal{Q}}|C\rangle\Big)[\hat{\mathcal{Q}}|P|B\rangle \\
& - \Big([E|\hat{P}|C\rangle[D|\hat{\mathcal{Q}}|F\rangle + [E|\hat{\mathcal{Q}}|C\rangle[D|\hat{P}|F\rangle\Big)[\hat{\mathcal{Q}}|P|B\rangle \\
& - [E|\hat{\mathcal{Q}}|B\rangle\Big([D|\hat{P}|C\rangle - [D|\hat{\mathcal{Q}}|C\rangle\Big)[\hat{Q}|P|F\rangle \\
& - [D|\hat{Q}|B\rangle\Big([E|\hat{P}|F\rangle - [E|\hat{Q}|F\rangle\Big)[\hat{\mathcal{Q}}|P|C\rangle \Big),
\end{aligned} \tag{3.42}$$

we find that this form gives a contribution to the bubble coefficient of,

$$G_1^{1:\hat{Q}} = -8\frac{\sqrt{\Delta_3}\beta + P.\mathcal{Q}\gamma}{P^2\Delta_3}. \tag{3.43}$$

Again $\mathcal{G}_1^{1:\hat{P}}$ and $G_1^{1:\hat{P}}$ are obtained by irrational conjugation.

Although these forms contain $\sqrt{\Delta_3}$, we are always interested in the sum of irrational-conjugate pairs and the final canonical form is guaranteed to be rational.

4 Example: seven-point $\mathcal{N}=1$ contributions

The basic NMHV amplitudes are:

$$
\begin{aligned}
A &: A_7(1^-,2^-,3^+,4^-,5^+,6^+,7^+) \\
B &: A_7(1^-,2^-,3^+,4^+,5^-,6^+,7^+) \\
C &: A_7(1^-,2^+,3^-,4^+,5^-,6^+,7^+) \\
D &: A_7(1^-,2^-,3^-,4^+,5^+,6^+,7^+)
\end{aligned}
\tag{4.1}
$$

Amplitude A has no permutation symmetries while amplitudes B and C have the following invariances:

$$
\begin{aligned}
B &: (1234567) \leftrightarrow (2176543) \\
C &: (1234567) \leftrightarrow (5432176)
\end{aligned}
\tag{4.2}
$$

Amplitude D has the simplest structure and is an example of a "split-helicity" amplitude whose n-point expression is given in [24].

The supersymmetric cancellations present in a $\mathcal{N}=1$ computation lead to the one-loop amplitude being of the form,

$$
A_n^{\mathcal{N}=1} = \sum_{i \in \mathcal{C}} a_i \, I_4^i + \sum_{j \in \mathcal{D}} b_j \, I_3^j + \sum_{k \in \mathcal{E}} c_k \, I_2^k \ ,
\tag{4.3}
$$

with no further rational terms. Since all the coefficients can be evaluated from four-dimensional unitarity these contributions are termed "cut constructible".

For the contribution from the $\mathcal{N}=1$ chiral multiplet the coefficients of the box integral functions contain sufficient information to determine the coefficients of the one and two mass triangle functions. As discussed in the appendix A, we choose to absorb these triangles into the box functions, leaving a basis of *truncated* boxes $\mathcal{F}_4^i$, three mass triangles and bubble functions,

$$
A_n^{\mathcal{N}=1 \text{ chiral}} = \sum_{i \in \mathcal{C}} a_i \, \mathcal{F}_4^i + \sum_{j \in \mathcal{D}_{3m}} b_j \, I_3^{3m \, j} + \sum_{k \in \mathcal{E}} c_k \, I_2^k \ .
\tag{4.4}
$$

4.1 $A_7^{\mathcal{N}=1}(1^-,2^-,3^+,4^-,5^+,6^+,7^+)$

This amplitude can be decomposed into 20 integral functions:

$$
\begin{aligned}
A_7^{\mathcal{N}=1 \text{ chiral}}(1^-,2^-,3^+,&4^-,5^+,6^+,7^+) = a_1^A \mathcal{F}_{6\{71\}\{23\}\{45\}}^{3m} \\
&+ a_2^A \mathcal{F}_{1\{23\}\{456\}7}^{2m\,h} + a_3^A \mathcal{F}_{3\{45\}\{671\}2}^{2m\,h} + a_4^A \mathcal{F}_{3\{456\}\{71\}2}^{2m\,h} + a_5^A \mathcal{F}_{5\{671\}\{23\}4}^{2m\,h} \\
&+ a_6^A \mathcal{F}_{3\{45\}6\{712\}}^{2m\,e} + a_7^A \mathcal{F}_{234\{5671\}}^{1m} + a_8^A \mathcal{F}_{345\{6712\}}^{1m} \\
&+ b_1^A I_{\{23\}\{45\}\{671\}}^{3m\,tri} + b_2^A I_{\{71\}\{23\}\{456\}}^{3m\,tri} \\
&+ c_1^A I_2(t_{123}) + c_2^A I_2(t_{234}) + c_3^A I_2(t_{345}) + c_4^A I_2(t_{456}) + c_6^A I_2(t_{671}) \\
&+ c_7^A I_2(t_{712}) \\
&+ d_2^A I_2(s_{23}) + d_3^A I_2(s_{34}) + d_4^A I_2(s_{45}) + d_7^A I_2(s_{71}),
\end{aligned}
\tag{4.5}
$$

where we have chosen to label the boxes and three-mass triangles by the clustering of the legs.

4.2 $A_7^{\mathcal{N}=1}(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+)$

This amplitude can be decomposed into 25 integral functions:

$$
\begin{aligned}
A_7^{\mathcal{N}=1 \text{ chiral}}(1^-, 2^-, 3^+, 4^+, 5^-, 6^+, 7^+) = {} & a_1^B \mathcal{F}_{6\{71\}\{23\}\{45\}}^{3m} + a_2^B \mathcal{F}_{4\{56\}\{71\}\{23\}}^{3m} \\
& + a_3^B \mathcal{F}_{1\{23\}\{456\}7}^{2m\,h} + a_4^B \mathcal{F}_{3\{45\}\{671\}2}^{2m\,h} + a_5^B \mathcal{F}_{6\{71\}\{234\}5}^{2m\,h} + a_6^B \mathcal{F}_{1\{234\}\{56\}7}^{2m\,h} \\
& + a_7^B \mathcal{F}_{5\{671\}\{23\}4}^{2m\,h} + a_8^B \mathcal{F}_{3\{456\}\{71\}2}^{2m\,h} \\
& + a_9^B \mathcal{F}_{3\{45\}6\{712\}}^{2m\,e} + a_{10}^B \mathcal{F}_{4\{56\}7\{123\}}^{2m\,e} + a_{11}^B \mathcal{F}_{456\{7123\}}^{1m} \\
& + b_1^B I_{\{23\}\{45\}\{671\}}^{3m\,tri} + b_2^B I_{\{71\}\{23\}\{456\}}^{3m\,tri} + b_3^B I_{\{56\}\{71\}\{234\}}^{3m\,tri} \\
& + c_1^B I_2(t_{123}) + c_2^B I_2(t_{234}) + c_3^B I_2(t_{345}) + c_4^B I_2(t_{456}) + c_5^B I_2(t_{567}) \\
& + c_6^B I_2(t_{671}) + c_7^B I_2(t_{712}) \\
& + d_2^B I_2(s_{23}) + d_4^B I_2(s_{45}) + d_5^B I_2(s_{56}) + d_7^B I_2(s_{71}).
\end{aligned}
$$

(4.6)

4.3 $A_7^{\mathcal{N}=1}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+)$

This amplitude can be decomposed into 37 integral functions:

$$
\begin{aligned}
A_7^{\mathcal{N}=1 \text{ chiral}}(1^-, 2^+, 3^-, 4^+, 5^-, 6^+, 7^+) = {} & a_1^C \mathcal{F}_{6\{71\}\{23\}\{45\}}^{3m} + a_2^C \mathcal{F}_{7\{12\}\{34\}\{56\}}^{3m} \\
& + a_3^C \mathcal{F}_{1\{23\}\{456\}7}^{2m\,h} + a_4^C \mathcal{F}_{2\{34\}\{567\}1}^{2m\,h} + a_5^C \mathcal{F}_{3\{45\}\{671\}2}^{2m\,h} + a_6^C \mathcal{F}_{4\{56\}\{712\}3}^{2m\,h} + a_7^C \mathcal{F}_{6\{71\}\{234\}5}^{2m\,h} \\
& + a_8^C \mathcal{F}_{1\{234\}\{56\}7}^{2m\,h} + a_9^C \mathcal{F}_{3\{456\}\{71\}2}^{2m\,h} + a_{10}^C \mathcal{F}_{4\{567\}\{12\}3}^{2m\,h} + a_{11}^C \mathcal{F}_{5\{671\}\{23\}4}^{2m\,h} + a_{12}^C \mathcal{F}_{6\{712\}\{34\}5}^{2m\,h} \\
& + a_{13}^C \mathcal{F}_{4\{56\}7\{123\}}^{2m\,e} + a_{14}^C \mathcal{F}_{6\{71\}2\{345\}}^{2m\,e} \\
& + a_{15}^C \mathcal{F}_{123\{4567\}}^{1m} + a_{16}^C \mathcal{F}_{234\{5671\}}^{1m} + a_{17}^C \mathcal{F}_{345\{6712\}}^{1m} + a_{18}^C \mathcal{F}_{456\{7123\}}^{1m} + a_{19}^C \mathcal{F}_{712\{3456\}}^{1m} \\
& + b_1^C I_{\{12\}\{34\}\{567\}}^{3m\,tri} + b_2^C I_{\{712\}\{34\}\{56\}}^{3m\,tri} + b_3^C I_{\{71\}\{23\}\{456\}}^{3m\,tri} + b_4^C I_{\{71\}\{234\}\{56\}}^{3m\,tri} + b_5^C I_{\{671\}\{23\}\{45\}}^{3m\,tri} \\
& + c_1^C I_2(t_{123}) + c_2^C I_2(t_{234}) + c_3^C I_2(t_{345}) + c_4^C I_2(t_{456}) + c_5^C I_2(t_{567}) \\
& + c_6^C I_2(t_{671}) + c_7^C I_2(t_{712}) \\
& + d_1^C I_2(s_{12}) + d_2^C I_2(s_{23}) + d_3^C I_2(s_{34}) + d_4^C I_2(s_{45}) + d_5^C I_2(s_{56}) + d_7^C I_2(s_{71}).
\end{aligned}
$$

(4.7)

Altogether these NMHV $\mathcal{N}=1$ contributions are specified by 82 coefficients. The box and three-mass triangle coefficients are special cases of generic forms which are given in appendix C and section 6 respectively. The twenty c_i coefficients are either special cases of the generic C_0 function (C.8) or are given by one of four forms, C_X, specific to the seven-point case. The remaining parts of the amplitudes are given by the fourteen d_i^α functions, which are not all independent but can be expressed in terms of six D_X functions.

4.4 C_X functions

We illustrate the calculation of the C_X functions by considering an explicit realisation of the C_B function which arises as the coefficient of $I_2(t_{712})$ and is obtained by computing the t_{712} cut of $A_7(1, 2, 3^-, 4^+, 5^-, 6^+, 7)$, where precisely one of legs 7, 1 or 2 has negative helicity. We label this leg m_1.

The product of tree amplitudes generated by the cut is,

$$\sum_h A^{\text{tree}}(-\ell_1^h, 7, 1, 2, \ell_2^{-h}) \times A^{\text{tree}}(-\ell_2^h, 3^-, 4^+, 5^-, 6^+, \ell_1^{-h}), \quad (4.8)$$

where the summation is over the complex scalar and fermionic states of the $\mathcal{N} = 1$ chiral multiplet. The six point NMHV tree amplitude has three terms,

$$A^{\text{tree}}(-\ell_2^h, 3^-, 4^+, 5^-, 6^+, \ell_1^{-h}) = T_1^h + T_2^h + T_3^h, \quad (4.9)$$

where,

$$T_1^h = \frac{[\ell_2|P_{456}|5\rangle^2 [\ell_1|P_{456}|5\rangle^2}{t_{456}[\ell_1 \ell_2][\ell_2 3]\langle 4 5\rangle\langle 5 6\rangle[\ell_1|P_{456}|4][3|P_{456}|6\rangle} \left(-\frac{[\ell_2|P_{456}|5\rangle}{[\ell_1|P_{456}|5\rangle}\right)^h,$$

$$T_2^h = \frac{[6|P_{56\ell_1}|3\rangle^2(\langle \ell_2 3\rangle[6 \ell_1])^2}{t_{56\ell_1}\langle \ell_2 3\rangle\langle 3 4\rangle[5 6][6 \ell_1][5|P_{56\ell_1}|\ell_2\rangle[\ell_1|P_{56\ell_1}|4\rangle} \left(-\frac{[6|P_{56\ell_1}|3\rangle}{\langle \ell_2 3\rangle[6 \ell_1]}\right)^h, \quad (4.10)$$

$$T_3^h = \frac{[4|P_{345}|\ell_1\rangle^2 [4|P_{345}|\ell_2\rangle^2}{t_{345}\langle 6 \ell_1\rangle\langle \ell_1 \ell_2\rangle[3 4][4 5][3|P_{345}|6\rangle[5|P_{345}|\ell_2\rangle} \left(\frac{-[4|P_{345}|\ell_1\rangle}{[4|P_{345}|\ell_2\rangle}\right)^h,$$

with h denoting the helicity of the leg ℓ_2: $h = 0$ for a scalar and $h = \pm 1$ for a fermion.

Summing over the multiplet in eq. (4.8) leads to term by term cancellations which we can express as ρ factors multiplying each product of $h = 0$ terms, leading to a cut integrand of the form,

$$\frac{\langle m_1 \ell_1\rangle^2\langle m_1 \ell_2\rangle^2}{\langle 7 1\rangle\langle 1 2\rangle\langle 2 \ell_2\rangle\langle \ell_2 \ell_1\rangle\langle \ell_1 7\rangle} \times \left(T_1^0 \rho_1 + T_2^0 \rho_2 + T_3^0 \rho_3\right), \quad (4.11)$$

where the ρ factors are,

$$\rho_1 = \frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle m_1 \ell_2\rangle[\ell_2|P_{456}|5\rangle\langle m_1 \ell_1\rangle[\ell_1|P_{456}|5\rangle},$$

$$\rho_2 = \frac{\langle Y_{B2} \ell_2\rangle^2}{\langle m_1 \ell_2\rangle[6|P_{56\ell_1}|3\rangle\langle m_1 \ell_1\rangle\langle \ell_2 3\rangle[6 \ell_1]}, \quad (4.12)$$

$$\rho_3 = \frac{[4|P_{345}|m_1\rangle^2\langle \ell_1 \ell_2\rangle^2}{\langle m_1 \ell_1\rangle[4|P_{345}|\ell_1\rangle\langle m_1 \ell_2\rangle[4|P_{345}|\ell_2\rangle},$$

with,

$$|Y_{B2}\rangle = [6 5]\langle 5 3\rangle|m_1\rangle + [6 7]\langle m_1 3\rangle|7\rangle + [6 1]\langle m_1 3\rangle|1\rangle + [6 2]\langle m_1 3\rangle|2\rangle. \quad (4.13)$$

Consequently the contribution of the T_1 terms to the cut integrand is,

$$\frac{\langle m_1 \ell_1\rangle\langle m_1 \ell_2\rangle}{\langle 7 1\rangle\langle 1 2\rangle\langle 2 \ell_2\rangle\langle \ell_2 \ell_1\rangle\langle \ell_1 7\rangle} \times \frac{[\ell_2|P_{456}|5\rangle[\ell_1|P_{456}|5\rangle\langle m_1|P_{712}P_{456}|5\rangle^2}{t_{456}[\ell_1 \ell_2][\ell_2 3]\langle 4 5\rangle\langle 5 6\rangle[\ell_1|P_{456}|4][3|P_{456}|6\rangle}$$

$$= \frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle 7 1\rangle\langle 1 2\rangle\langle 4 5\rangle\langle 5 6\rangle[3|P_{456}|6\rangle t_{456}t_{712}} \times \frac{\langle m_1 \ell_1\rangle\langle m_1 \ell_2\rangle}{\langle 2 \ell_2\rangle\langle \ell_1 7\rangle} \times \frac{[\ell_2|P_{456}|5\rangle[\ell_1|P_{456}|5\rangle}{[\ell_2 3][\ell_1|P_{456}|4\rangle}$$

$$= \frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle 7 1\rangle\langle 1 2\rangle\langle 4 5\rangle\langle 5 6\rangle[3|P_{456}|6\rangle t_{456}t_{712}} \times \frac{\langle m_1 \ell_1\rangle\langle m_1 \ell_2\rangle}{\langle 2 \ell_2\rangle\langle \ell_1 7\rangle} \times \frac{\langle \ell_1|P_{712}P_{456}|5\rangle\langle \ell_2|P_{712}P_{456}|5\rangle}{\langle \ell_1|P_{712}|3]\langle \ell_2|P_{712}P_{456}|4\rangle}.$$

$$(4.14)$$

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This can be recognised as a $\mathcal{H}_4$ canonical form and thus yields a contribution to the bubble coefficient of,

$$-\frac{\langle m_1|P_{712}P_{456}|5\rangle^2}{\langle 7\,1\rangle\langle 1\,2\rangle\langle 4\,5\rangle\langle 5\,6\rangle[3|P_{456}|6\rangle t_{456}t_{712}}$$
$$\times H_4[2,7,P_{712}|3],P_{712}P_{456}|4\rangle;m_1,m_1,P_{712}P_{456}|5\rangle,P_{712}P_{456}|5\rangle;P_{712}]. \tag{4.15}$$

Similarly the T_2 and T_3 terms give a contribution,

$$\frac{1}{\langle 7\,1\rangle\langle 1\,2\rangle\langle 3\,4\rangle[5\,6]}G_4[2,7,P_{34}|5],P_{712}P_{56}|4\rangle;m_1,m_1,3,Y_{B2},Y_{B2};6;P_{34};P_{712}]$$
$$+\frac{[4|P_{456}|m_1\rangle^2}{\langle 7\,1\rangle\langle 1\,2\rangle[3\,4][4\,5][3|P_{345}|6\rangle t_{345}}H_4[2,7,6,P_{345}|5];m_1,m_1,P_{345}|4\rangle,P_{345}|4\rangle;P_{712}]. \tag{4.16}$$

We define the C_B function by generalising the result of this specific cut:

$$C_B[a,b,c,d,e,f,g;m_1]\equiv$$
$$-\frac{\langle m_1|P_{gab}P_{def}|e\rangle^2}{\langle g\,a\rangle\langle a\,b\rangle\langle d\,e\rangle\langle e\,f\rangle[c|P_{def}|f\rangle t_{def}t_{gab}}$$
$$\times H_4[b,g,P_{gab}|c],P_{gab}P_{def}|d\rangle;m_1,m_1,P_{gab}P_{def}|e\rangle,P_{gab}P_{def}|e\rangle;P_{gab}]$$
$$+\frac{1}{\langle g\,a\rangle\langle a\,b\rangle\langle c\,d\rangle[e\,f]}G_4[b,g,P_{cd}|e],P_{gab}P_{ef}|d\rangle;m_1,m_1,c,Y_{B2},Y_{B2};f;P_{cd};P_{gab}]$$
$$+\frac{[d|P_{cde}|m_1\rangle^2}{\langle g\,a\rangle\langle a\,b\rangle[c\,d][d\,e][c|P_{cde}|f\rangle t_{cde}}H_4[b,g,f,P_{cde}|e];m_1,m_1,P_{cde}|d],P_{cde}|d];P_{gab}], \tag{4.17}$$

with,

$$|Y_{B2}\rangle=[f\,e]\langle e\,c\rangle|m_1\rangle+[f\,g]\langle m_1\,c\rangle|g\rangle+[f\,a]\langle m_1\,c\rangle|a\rangle+[f\,b]\langle m_1\,c\rangle|b\rangle. \tag{4.18}$$

Five of the bubble coefficients can be expressed in terms of the C_B function:

$$c_7^C=C_B[1,2,3,4,5,6,7;1], \qquad c_6^C=-C_B[7,6,5,4,3,2,1;1],$$
$$c_5^C=C_B[6,7,1,2,3,4,5;5], \qquad c_4^C=-C_B[5,4,3,2,1,7,6;5], \tag{4.19}$$
$$c_6^A=C_B[7,1,2,3,4,5,6;1].$$

We define three further functions in this class, C_A, C_C and C_D;

$$C_A[a,b,c,d,e,f,g;m_1]$$
$$\equiv\frac{\langle m_1|P_{gab}P_{cd}|e\rangle^2}{\langle c\,d\rangle\langle d\,e\rangle[f|P_{de}|c\rangle\langle g\,a\rangle\langle a\,b\rangle t_{cde}t_{gab}}H_3[b,g,P_{gab}|f];m_1,m_1,P_{gab}P_{cd}|e\rangle;P_{gab}] \tag{4.20}$$
$$-\frac{[d|P_{ef}|m_1\rangle^2}{\langle g\,a\rangle\langle a\,b\rangle[d\,e][e\,f][f|P_{de}|c\rangle t_{def}}H_3[b,c,g;m_1,m_1,P_{ef}|d];P_{gab}],$$
$$C_C[a,b,c,d,e,f,g;m_1]$$
$$\equiv\frac{[d\,e]^3\langle m_1\,f\rangle^2}{t_{cde}[c\,d][c|P_{de}|f\rangle\langle g\,a\rangle\langle a\,b\rangle}H_3[g,b,P_{cd}|e];f,m_1,m_1;P_{gab}] \tag{4.21}$$

$$- \frac{\langle m_1 | P_{gab} P_{de} | f \rangle^2}{t_{def} t_{gab} \langle d\,e \rangle \langle e\,f \rangle [c | P_{de} | f \rangle \langle g\,a \rangle \langle a\,b \rangle}$$
$$\times \overline{H}_4[c, P_{ef}|d\rangle, P_{ga}|b\rangle, P_{ab}|g\rangle; P_{de}|f\rangle, P_{de}|f\rangle, P_{gab}|m_1\rangle, P_{gab}|m_1\rangle; P_{gab}]$$
$$+ \frac{1}{[e\,f]\langle c\,d \rangle \langle g\,a \rangle \langle a\,b \rangle}$$
$$\times G_5[P_{cd}|e], P_{gab}P_{ef}|d\rangle, P_{gab}|f], b, g; c, m_1, m_1, P_{gab}|e], Y_C, Y_C; e; P_{cd}; P_{gab}],$$

$$C_D[a, b, c, d, e, f, g; m_1]$$
$$\equiv \frac{\langle d\,e \rangle^3 [f | P_{gab} | m_1 \rangle^2}{t_{cde} t_{gab} \langle c\,d \rangle [f | P_{de} | c \rangle \langle g\,a \rangle \langle a\,b \rangle} H_3[b, g, P_{gab} P_{cd}|e\rangle; m_1, m_1, P_{gab}|f]; P_{gab}] \quad (4.22)$$
$$- \frac{[f | P_{de} | m_1 \rangle^2}{t_{def} [d\,e][e\,f][f | P_{de} | c \rangle \langle g\,a \rangle \langle a\,b \rangle} H_4[b, c, g, P_{ef}|d]; m_1, m_1, P_{de}|f], P_{de}|f]; P_{gab}]$$
$$- \frac{1}{\langle e\,f \rangle [c\,d] \langle g\,a \rangle \langle a\,b \rangle} G_5[f, b, g, P_{ef}|d], P_{gab} P_{cd}|e\rangle; e, e, m_1, m_1, Y_D, Y_D; c; P_{ef}; P_{gab}],$$

where,

$$|Y_C\rangle = [e\,f] \langle f\,c \rangle | m_1 \rangle + \langle m_1 c \rangle ([e\,g]|g\rangle + [e\,a]|a\rangle + [e\,b]|b\rangle),$$
$$|Y_D\rangle = - [c\,d] \langle d\,e \rangle | m_1 \rangle + \langle e\,m_1 \rangle ([c\,g]|g\rangle + [c\,a]|a\rangle + [c\,b]|b\rangle). \quad (4.23)$$

The remaining c_i^X coefficients are then:

$$\begin{aligned}
c_1^A &= C_0[2,3,4,5,6,7,1;3,4], & c_2^A &= C_0[3,4,5,6,7,1,2;3,1], \\
c_3^A &= C_A[4,5,6,7,1,2,3;4], & c_4^A &= C_D[5,6,7,1,2,3,4;4], \\
c_7^A &= C_0[1,2,3,4,5,6,7;7,4], & & \\
c_1^B &= C_0[2,3,4,5,6,7,1;3,5], & c_2^B &= C_C[3,4,5,6,7,1,2;2], \\
c_3^B &= C_A[4,5,6,7,1,2,3;5], & c_4^B &= C_D[5,6,7,1,2,3,4;5], \quad (4.24) \\
c_5^B &= -C_A[6,5,4,3,2,1,7;5], & c_6^B &= -C_C[7,6,5,4,3,2,1;1], \\
c_7^B &= -C_0[1,7,6,5,4,3,2;7,5], & & \\
c_1^C &= C_0[2,3,4,5,6,7,1;5,2], & c_2^C &= C_C[3,4,5,6,7,1,2;3], \\
c_3^C &= C_0[4,5,6,7,1,2,3;4,1]. & &
\end{aligned}$$

4.5 D_X functions

The D_X functions arise as the coefficients of $I_2(s_{ab})$ when we consider cuts of the form,

$$\int d\text{LIPS}\, A^{\text{tree}}(-\ell_1, a, b, \ell_2) \times A^{\text{tree}}(-\ell_2, c, d, e, f, g, \ell_1) \to D[a, b, c, d, e, f, g]. \quad (4.25)$$

In order to obtain non-vanishing $\mathcal{N} = 1$ and scalar contributions, legs a and b must be of opposite helicity. For the $\mathcal{N} = 1$ contribution the two helicity configurations for a and b are trivially related:

$$D[a^+, b^-, c, d, e, f, g] = -D[b^-, a^+, c, d, e, f, g]. \quad (4.26)$$

Consequently, the number of independent D_X functions corresponds to the number of independent helicity configurations for the legs c, d, e, f, g that contain two negative and

three positive helicities. There are six such configurations: $(--+++)$, $(-+-++)$, $(-++-+)$, $(-+++-)$, $(+--++)$ and $(+-+-+)$.

To evaluate these cuts we need the explicit forms of the seven-point NMHV tree amplitudes where two external states are scalars or fermions (given explicitly in appendix D). Using these forms for the tree amplitude $A^{\text{tree}}(\ell_2^h, c^-, d^-, e^+, f^+, g^+, \ell_1^{-h})$, we express the cut of the first of these helicity configurations in terms of canonical forms and obtain,

$$
\begin{aligned}
&D_A[a,b,c,d,e,f,g] \\
&= \frac{[e|P_{cde}|a\rangle^2}{\langle a\,b\rangle\langle f\,g\rangle[c\,d][d\,e][c|P_{cde}|f\rangle t_{cde}} H_2[b,g;a,P_{cde}|e];P_{ab}] \\
&\quad - \frac{\langle f\,d\rangle}{[a\,b]\langle d\,e\rangle\langle e\,f\rangle\langle f\,g\rangle[c|P_{de}|f\rangle} \\
&\quad \times \overline{H}_4[a,c,P_{cde}|f\rangle, P_{ab}P_{abc}P_{de}|f\rangle);b,g,X_A,X_A;P_{ab}] \\
&\quad - \frac{1}{[a\,b]\langle d\,e\rangle\langle e\,f\rangle\langle f\,g\rangle[c|P_{de}|f\rangle} \\
&\quad \times \overline{H}_5[a,P_{ab}|g\rangle,c,P_{cde}|f\rangle, P_{ab}P_{abc}P_{de}|f\rangle);b,P_{ab}|f\rangle,P_{ef}|d\rangle,X_A,X_A;P_{ab}] \\
&\quad - \frac{\langle c\,d\rangle^4[g\,b]^2}{[a\,b]\langle c\,d\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{ab}|c\rangle t_{gab}} H_2[b,P_{ab}P_{gab}|f\rangle;a,P_{ab}|g];P_{ab}] \\
&\quad - \frac{\langle a\,c\rangle^2[g|P_{abc}|d\rangle^4}{\langle a\,b\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{ab}|c\rangle[g|P_{abc}|d\rangle t_{abc}t_{def}} H_2[b,P_{abc}P_{de}|f\rangle;a,c;P_{ab}],
\end{aligned}
\tag{4.27}
$$

where,

$$
\begin{aligned}
|X_A] =& \langle f\,d\rangle[b\,a]\langle a\,g\rangle|g] + \langle f\,d\rangle[b\,a]\langle a\,f\rangle|f] - \langle a\,f\rangle[b\,a]\langle d\,e\rangle|e] \\
& + (\langle f\,g\rangle[g\,c]\langle c\,d\rangle + [c|P_{abc}|f\rangle\langle c\,d\rangle + \langle f\,d\rangle s_{ab})|b].
\end{aligned}
\tag{4.28}
$$

For the other helicity configurations we define $D_{B,C,D,E,F}$ in a similar fashion. The explicit forms of these are given in appendix E. In terms of these we have,

$$
\begin{aligned}
d_2^A &= D_D[2,3,4,5,6,7,1], & d_3^A &= -D_A[4,3,2,1,7,6,5], \\
d_4^A &= -D_E[5,4,3,2,1,7,6], & d_7^A &= D_B[7,1,2,3,4,5,6], \\
d_2^B &= -D_C[2,1,7,6,5,4,3], & d_4^B &= D_E[4,5,3,2,1,7,6], \\
d_5^B &= -D_E[6,5,7,1,2,3,4], & d_7^B &= D_C[1,2,3,4,5,6,7], \\
d_1^C &= -D_B[2,1,3,4,5,6,7], & d_2^C &= D_C[3,1,7,6,5,4,2], \\
d_3^C &= -D_C[3,5,6,7,1,2,4], & d_4^C &= D_B[4,5,3,2,1,7,6], \\
d_5^C &= D_F[5,6,7,1,2,3,4], & d_7^C &= -D_F[1,7,6,5,4,3,2].
\end{aligned}
\tag{4.29}
$$

For each amplitude we have checked at explicit kinematic points that these bubble coefficients satisfy,

$$
\sum_i c_i + \sum_j d_j = A^{\text{tree}}.
\tag{4.30}
$$

These conditions ensure that each amplitude has the correct $1/\epsilon$ IR singularity [38].
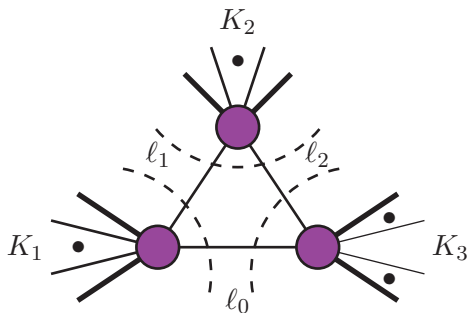
Figure 2. The triple cut of an amplitude

5 Canonical basis for triangle coefficients from triple cuts

Generalisations of unitarity can be used to determine the coefficients of triangle and box functions simply. If we consider a triple cut,

$$C_3 = -\int d^4\ell\delta(\ell_0^2)\delta(\ell_1^2)\delta(\ell_2^2) \sum A\Big((-\ell_0), m, \dots, j-1, (\ell_1)\Big) \\ \times A\Big((-\ell_1), j, \dots, l-1, (\ell_2)\Big) \times A\Big((-\ell_2), l, \dots, m-1, (\ell_0)\Big),$$

(5.1)

The minus sign in this equation is for when using colour-ordered partial amplitudes as normalised in eq. (2.1). This has contributions from the discontinuities of a single triangle and several box functions,

$$C_3 = \left(\sum_{i \in \mathcal{C}''} a_i\, I_4^i + b_j\, I_3^j\right)\Big|_{\text{Disc}}.$$

(5.2)

The information in this cut may be used to determine the triangle coefficient b_j.

One of the advantages of the supersymmetric decomposition of gluonic amplitudes is that the one and two-mass triangle coefficients need not be explicitly computed: for the $\mathcal{N} = 4$ contributions they are absent while for the $\mathcal{N} = 1$ and scalar contributions they are tied to the box coefficients and can be incorporated into the truncated box functions. Consequently this section will focus on the case where all three masses of the triangle are non-zero. Consider a physical triple cut in an amplitude where all three corners are massive as in figure 2.

As the momentum invariants, $K_1 \equiv k_m + k_{m+1} + \cdots + k_{j-1}$ etc, are all non-null, there exist kinematic regimes is which the integration has non-vanishing support for real loop momentum.

In this section we present the contributions of various canonical forms to the coefficient of the three-mass triangle integral function. As before we build our canonical forms from a simple starting point. Consider,

$$\mathcal{E}_1(a; b) \equiv \frac{\langle \ell_0\, b\rangle}{\langle \ell_0\, a\rangle} = \frac{[a|\ell_0|b\rangle}{(\ell_0 + k_a)^2}.$$

(5.3)

We chose to manipulate this as if it were a covariant integral. This means effectively replacing,

$$\int d\text{LIPS} \longrightarrow \int \frac{d^D\ell}{\ell_0^2\ell_1^2\ell_2^2},$$

(5.4)
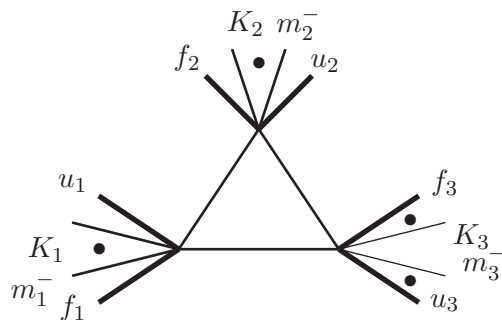
Figure 3. The three mass triangle integral function appearing in the NMHV amplitudes indicating the labeling of the legs.

then evaluating the covariant integral only keeping the coefficient of the three-mass triangle in the result.

The covariant integral is a linear box function. Evaluating this we find the following contribution to the three-mass triangle coefficient:

$$E_1[a;b] = -\frac{\langle a|[K_1, K_2]|b\rangle}{2\langle a|K_1 K_2|a\rangle}. \tag{5.5}$$

As in eq. (3.22), we could write the denominator as,

$$\langle a|\hat{K}_1 \hat{K}_2|a\rangle = \langle a\,\hat{K}_1\rangle[\hat{K}_1\,\hat{K}_2]\langle \hat{K}_2\,a\rangle, \tag{5.6}$$

where $\hat{K}_i$ are the null linear combinations of K_1 and K_2. These linear combinations involve irrational coefficients as in eq. (3.23). The case where $\langle a\,\hat{K}_i\rangle = 0$ must be treated as a special case. This does not arise when a denotes an external momentum but may arise in the derivations of more complicated canonical forms. When $a = \hat{K}_3$, we have,

$$\mathcal{E}_1^x = \frac{\langle b\,\ell\rangle}{\langle \hat{K}_3\,\ell\rangle} \longrightarrow E_1^x[K_3, b] = -\frac{\langle b|K_1|K_3]}{2\langle K_3|K_1|K_3]}. \tag{5.7}$$

More complicated forms can readily be generated from this simple starting point. A summary of these canonical forms is given in appendix B. As an example of the use of these canonical forms, the general expression for a NMHV three-mass triangle coefficient is given in the next section.

6 Example: n-point three-mass triangles for $\mathcal{N} = 1$ and scalar contributions to NMHV amplitudes

As an example of using the canonical forms for the three mass triangle let us evaluate the general form for the three-mass triangle for NMHV amplitudes. The $\mathcal{N} = 1$ contribution was previously presented in [9].

For both the chiral $\mathcal{N} = 1$ and scalar contributions, the only three-mass triangles which appear in the NMHV amplitude have exactly one negative helicity on each corner. Consider such an integral function with the following labelling as shown in figure 3.

First we take the $\mathcal{N} = 1$ case. Evaluating this using the triple cut of figure 2 the triple cut integrand is,

$$\sum_h A(-\ell_0^h, \ldots, m_1^-, \ldots, \ell_1^{-h}) \times A(-\ell_1^h, \ldots, m_2^-, \ldots, \ell_2^{-h}) \times A(-\ell_2^h, \ldots, m_3^-, \ldots, \ell_0^{-h}), \quad (6.1)$$

where $\ell_1 = \ell_0 - K_1$ etc. The summation is over the $\mathcal{N} = 1$ chiral multiplet. The effect of this summation is to give the scalar contribution times a ρ-factor,

$$A(-\ell_0^0, \ldots, m_1^-, \ldots, \ell_1^0) \times A(-\ell_1^0, \ldots, m_2^-, \ldots, \ell_2^0) \times A(-\ell_2^0, \ldots, m_3^-, \ldots, \ell_0^0) \times \rho, \quad (6.2)$$

where,

$$\begin{aligned} \rho &= -\frac{(\langle m_1\, \ell_1\rangle \langle m_2\, \ell_2\rangle \langle m_3\, \ell_0\rangle - \langle m_1\, \ell_0\rangle \langle m_2\, \ell_1\rangle \langle m_3\, \ell_2\rangle)^2}{\langle m_1\, \ell_1\rangle \langle m_2\, \ell_2\rangle \langle m_3\, \ell_0\rangle \langle m_1\, \ell_0\rangle \langle m_2\, \ell_1\rangle \langle m_3\, \ell_2\rangle} \\ &= -\frac{\langle \ell_0\, X\rangle^2}{[\ell_1\, \ell_2]^2 \langle m_1\, \ell_1\rangle \langle m_2\, \ell_2\rangle \langle m_3\, \ell_0\rangle \langle m_1\, \ell_0\rangle \langle m_2\, \ell_1\rangle \langle m_3\, \ell_2\rangle}, \end{aligned} \quad (6.3)$$

and,

$$|X\rangle = |m_1\rangle \langle m_3|K_3 K_2|m_2] + |m_3\rangle \langle m_1|K_1 K_2|m_2]. \quad (6.4)$$

The cut is then,

$$\begin{aligned} &-\frac{\langle m_1\, \ell_0\rangle \langle m_1\, \ell_1\rangle}{\langle \ell_0\, f_1\rangle \langle f_1 \cdots u_1\rangle \langle u_1\, \ell_1\rangle \langle \ell_1\, \ell_0\rangle} \times \frac{\langle m_2\, \ell_1\rangle \langle m_2\, \ell_2\rangle}{\langle \ell_1\, f_2\rangle \langle f_2 \cdots u_2\rangle \langle u_2\, \ell_2\rangle \langle \ell_2\, \ell_1\rangle} \\ &\qquad\qquad\qquad \times \frac{\langle m_3\, \ell_2\rangle \langle m_3\, \ell_0\rangle}{\langle \ell_2\, f_3\rangle \langle f_3 \cdots u_3\rangle \langle u_3\, \ell_0\rangle \langle \ell_0\, \ell_2\rangle} \times \frac{\langle \ell_0\, X\rangle^2}{[\ell_1\, \ell_2]^2} \\ &= C_0 \times \frac{\langle m_1\, \ell_0\rangle \langle m_1\, \ell_1\rangle}{\langle \ell_0\, f_1\rangle \langle u_1\, \ell_1\rangle} \times \frac{\langle m_2\, \ell_1\rangle \langle m_2\, \ell_2\rangle}{\langle \ell_1\, f_2\rangle \langle f_2\, \ell_2\rangle} \times \frac{\langle m_3\, \ell_2\rangle \langle m_3\, \ell_0\rangle}{\langle \ell_2\, f_3\rangle \langle u_3\, \ell_0\rangle} \times \frac{\langle X\, \ell_0\rangle^2}{\langle \ell_0|\ell_1 \ell_2|\ell_0\rangle}, \end{aligned} \quad (6.5)$$

where,

$$C_0 = -\frac{\langle u_1\, f_2\rangle \langle u_2\, f_3\rangle \langle u_3\, f_1\rangle}{\prod_i \langle i\, i+1\rangle K_2^2}. \quad (6.6)$$

This can be turned into a function of ℓ_0 only:

$$C_0 \times \frac{\prod_{|y\rangle \in T_1} \langle \ell_0\, y\rangle}{\prod_{|x\rangle \in S} \langle \ell_0\, x\rangle} \times \frac{1}{\langle \ell_0|K_1 K_3|\ell_0\rangle}, \quad (6.7)$$

where,

$$\begin{aligned} S &= \{|f_1\rangle, |u_3\rangle, K_3 K_2|f_2\rangle, K_3 K_2|u_1\rangle, K_1 K_2|f_3\rangle, K_1 K_2|u_2\rangle\}, \\ T_1 &= \{|m_1\rangle, |m_3\rangle, K_3 K_2|m_1\rangle, K_3 K_2|m_2\rangle, K_1 K_2|m_2\rangle, K_1 K_2|m_3\rangle, |X\rangle, |X\rangle\}, \end{aligned} \quad (6.8)$$

and we have used,

$$\frac{\langle \ell_1\, a\rangle}{\langle \ell_1\, b\rangle} = \frac{\langle \ell_0|\ell_2 \ell_1|a\rangle}{\langle \ell_0|\ell_2 \ell_1|b\rangle} = \frac{\langle \ell_0|K_3 K_2|a\rangle}{\langle \ell_0|K_3 K_2|b\rangle}, \qquad \frac{\langle \ell_2\, a\rangle}{\langle \ell_2\, b\rangle} = \frac{\langle \ell_0|K_1 K_2|a\rangle}{\langle \ell_0|K_1 K_2|b\rangle}. \quad (6.9)$$

This is precisely the canonical form $\mathcal{J}_n^0$ with $n = 6$ as defined in appendix B. So the three mass triangle coefficient is precisely,

$$b_3^{3m} = C_0 \times J_6^0(S; T_1; K_i). \quad (6.10)$$

This general expression simplifies in many cases: if the m_i coincide with any of the u_i or f_i, the J_6^0 function simplifies to J_n^0 with $n < 6$.

The scalar case is more complicated. The cut integrand is,

$$
\frac{\langle m_1\,\ell_0\rangle^2\langle m_1\,\ell_1\rangle^2}{\langle \ell_0\,f_1\rangle\langle f_1\cdots u_1\rangle\langle u_1\,\ell_1\rangle\langle \ell_1\,\ell_0\rangle} \times \frac{\langle m_2\,\ell_1\rangle^2\langle m_2\,\ell_2\rangle^2}{\langle \ell_1\,f_2\rangle\langle f_2\cdots u_2\rangle\langle u_2\,\ell_2\rangle\langle \ell_2\,\ell_1\rangle}
$$

$$
\times \frac{\langle m_3\,\ell_2\rangle^2\langle m_3\,\ell_0\rangle^2}{\langle \ell_2\,f_3\rangle\langle f_3\cdots u_3\rangle\langle u_3\,\ell_0\rangle\langle \ell_0\,\ell_2\rangle}
$$

$$
= C_0 \times \frac{\langle m_1\,\ell_0\rangle^2\langle m_1\,\ell_1\rangle^2}{\langle \ell_0\,f_1\rangle\langle u_1\,\ell_1\rangle} \times \frac{\langle m_2\,\ell_1\rangle^2\langle m_2\,\ell_2\rangle^2}{\langle \ell_1\,f_2\rangle\langle f_2\,\ell_2\rangle} \times \frac{\langle m_3\,\ell_2\rangle^2\langle m_3\,\ell_0\rangle^2}{\langle \ell_2\,f_3\rangle\langle u_3\,\ell_0\rangle} \times \frac{[\ell_1\,\ell_2]^2}{\langle \ell_0|\ell_1\ell_2|\ell_0\rangle}
$$

$$
= C_0 \times \frac{\langle m_1\,\ell_0\rangle^2\langle m_3\,\ell_0\rangle^2}{\langle \ell_0\,f_1\rangle\langle u_1\,\ell_0\rangle} \times \frac{\langle m_1\,\ell_1\rangle^2}{\langle \ell_1\,f_2\rangle\langle u_2\,\ell_1\rangle} \times \frac{\langle m_3\,\ell_2\rangle^2}{\langle f_3\,\ell_2\rangle\langle \ell_2\,f_3\rangle} \times \frac{\langle m_2\,\ell_2\rangle^2[\ell_1\,\ell_2]^2\langle m_2\,\ell_1\rangle^2}{\langle \ell_0|K_2K_3|\ell_0\rangle}
$$

$$
= C_0 \times \frac{\prod_{|y\rangle\in T_2}\langle \ell_0\,y\rangle}{\prod_{|x\rangle\in S}\langle \ell_0\,x\rangle} \times \frac{\langle m_2\,\ell_2\rangle^2[\ell_1\,\ell_2]^2\langle m_2\,\ell_1\rangle^2}{\langle \ell_0|K_2K_3|\ell_0\rangle},
$$

(6.11)

where,

$$
T_2 = \{|m_1\rangle, |m_1\rangle, |m_3\rangle, |m_3\rangle, K_1K_2|m_3\rangle, K_1K_2|m_3\rangle, K_3K_2|m_1\rangle, K_3K_2|m_1\rangle\}. \tag{6.12}
$$

Splitting the pole and defining $T' = T_2 - \{|m_1\rangle, |m_1\rangle, |m_3\rangle\}$, the cut integrand becomes,

$$
C_0\frac{\prod_{|y\rangle\in T_2}\langle \ell_0\,y\rangle}{\prod_{|x\rangle\in S}\langle \ell_0\,x\rangle} \times \frac{\langle m_2|\ell_1\ell_2|m_2\rangle^2}{\langle \ell_0|K_2K_3|\ell_0\rangle}
$$

$$
= C_0 \sum_{|x\rangle\in S} C_x \frac{\langle \ell_0\,m_1\rangle\langle \ell_0\,m_1\rangle\langle \ell_0\,m_3\rangle}{\langle \ell_0\,x\rangle} \times \frac{(\langle m_2|K_1K_3|m_2\rangle + \langle m_2|\ell_0K_2|m_2\rangle)^2}{\langle \ell_0|K_2K_3|\ell_0\rangle}
$$

$$
\rightarrow C_0 \sum_{|x\rangle\in S} C_x \Big(\langle m_2|K_1K_3|m_2\rangle^2 J_1^0(x;m_1,m_1,m_3)
$$

$$
+ 2\langle m_2|K_1K_3|m_2\rangle J_1^1(x;m_2,m_1,m_1,m_3;K_2|m_2\rangle)
$$

$$
+ J_1^2(x;m_2,m_2,m_1,m_1,m_3;K_2|m_2\rangle,K_2|m_2\rangle) \Big),
$$

(6.13)

where,

$$
C_x = \frac{\prod_{|y\rangle\in T'}\langle y\,x\rangle}{\prod_{|z\rangle\in S-\{|x\rangle\}}\langle z\,x\rangle}. \tag{6.14}
$$

This is a general, explicit, rational form of the n-point NMHV three-mass triangle coefficient.

7 Summary

We have presented an implementation of the Unitarity method and applied it to the computation of the seven-point one-loop $\mathcal{N} = 1$ amplitudes. Once the canonical forms are derived, the method is algebraic. In many ways our canonical form approach is equivalent to alternate methods however it naturally applies to trees written in the spinor helicity formalism and produces results which are manifestly rational.

Acknowledgments

A   Integral functions

The scalar box integral is,

$$
I_4 = -i \, (4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \, \frac{1}{p^2 \, (p-K_1)^2 \, (p-K_1-K_2)^2 \, (p+K_4)^2} \, , \tag{A.1}
$$

where K_i is the sum of the momenta of the legs attached to the i-th corner. If a single leg is attached then K_i is null. The form of the integral depends upon the number of the K_i which are non-null, $K_i^2 \neq 0$. We often misname these *massive legs*. The integrals are functions of the non-zero K_i^2 and the invariants,

$$
S \equiv (K_1+K_2)^2, \quad T \equiv (K_2+K_3)^2. \tag{A.2}
$$

For convenience, we always define these integrals taking leg 1 as massless and leg 4 as massive.

It is convenient to define the scalar box function, F,

$$
F(K_1,K_2,K_3,K_4) = -\frac{2\sqrt{\det S}}{r_\Gamma} \, I_4, \tag{A.3}
$$

where,

$$
r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{A.4}
$$

and the symmetric 4×4 matrix S has components (i, j are mod4),

$$
S_{ij} = -\frac{1}{2} \left(K_i + \cdots + K_{j-1} \right)^2, \quad i \neq j; \qquad S_{ii} = 0. \tag{A.5}
$$

In terms of these variables, the relationships between the scalar box functions and scalar box integrals are given by,

$$
\begin{aligned}
I_4^{1\mathrm{m}} &= -2r_\Gamma \frac{F^{1\mathrm{m}}}{ST} \, , & I_4^{2\mathrm{m}e} &= -2r_\Gamma \frac{F^{2\mathrm{m}\,e}}{ST - K_2^2 K_4^2} \, , \\
I_4^{2\mathrm{m}h} &= -2r_\Gamma \frac{F^{2\mathrm{m}\,h}}{ST}, & I_4^{3\mathrm{m}} &= -2r_\Gamma \frac{F^{3\mathrm{m}}}{ST - K_2^2 K_4^2} \, .
\end{aligned} \tag{A.6}
$$

With the labelling of legs shown in figure 4, the scalar box functions expanded to $\mathcal{O}(\epsilon^0)$ in the different cases reduce to,

$$
\begin{aligned}
F^{1\mathrm{m}} = {}& -\frac{1}{\epsilon^2} \Big[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_4^2)^{-\epsilon} \Big] \\
& + \mathrm{Li}_2 \left(1 - \frac{K_4^2}{S} \right) \, + \, \mathrm{Li}_2 \left(1 - \frac{K_4^2}{T} \right) + \frac{1}{2} \ln^2 \left(\frac{S}{T} \right) \, + \, \frac{\pi^2}{6} \, ,
\end{aligned} \tag{A.7}
$$
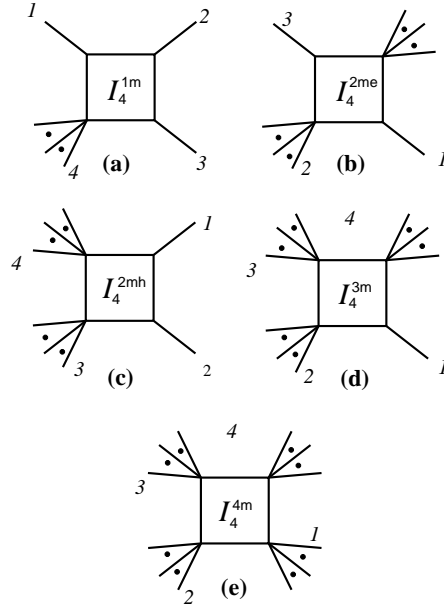
Figure 4. The different types of scalar box.

$$
\begin{aligned}
F^{2\mathrm{m}\,e} = & -\frac{1}{\epsilon^2}\Big[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_2^2)^{-\epsilon} - (-K_4^2)^{-\epsilon}\Big] \\
& + \mathrm{Li}_2\left(1 - \frac{K_2^2}{S}\right) + \mathrm{Li}_2\left(1 - \frac{K_2^2}{T}\right) + \mathrm{Li}_2\left(1 - \frac{K_4^2}{S}\right) \\
& + \mathrm{Li}_2\left(1 - \frac{K_4^2}{T}\right) - \mathrm{Li}_2\left(1 - \frac{K_2^2 K_4^2}{ST}\right) + \frac{1}{2}\ln^2(S/T) \,,
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
F^{2\mathrm{m}\,h} = & -\frac{1}{\epsilon^2}\Big[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_3^2)^{-\epsilon} - (-K_4^2)^{-\epsilon}\Big] \\
& - \frac{1}{2\epsilon^2}\frac{(-K_3^2)^{-\epsilon}(-K_4^2)^{-\epsilon}}{(-S)^{-\epsilon}} + \frac{1}{2}\ln^2(S/T) \\
& + \mathrm{Li}_2\left(1 - \frac{K_3^2}{T}\right) + \mathrm{Li}_2\left(1 - \frac{K_4^2}{T}\right) \,,
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
F^{3\mathrm{m}} = & -\frac{1}{\epsilon^2}\Big[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_2^2)^{-\epsilon} - (-K_3^2)^{-\epsilon} - (-K_4^2)^{-\epsilon}\Big] \\
& - \frac{1}{2\epsilon^2}\frac{(-K_2^2)^{-\epsilon}(-K_3^2)^{-\epsilon}}{(-T)^{-\epsilon}} - \frac{1}{2\epsilon^2}\frac{(-K_3^2)^{-\epsilon}(-K_4^2)^{-\epsilon}}{(-S)^{-\epsilon}} + \frac{1}{2}\ln^2\left(\frac{S}{T}\right) \\
& + \mathrm{Li}_2\left(1 - \frac{K_2^2}{S}\right) + \mathrm{Li}_2\left(1 - \frac{K_4^2}{T}\right) - \mathrm{Li}_2\left(1 - \frac{K_2^2 K_4^2}{ST}\right) \,.
\end{aligned}
\tag{A.10}
$$

Our seven-point expressions do not need the explicit form of the four mass scalar box.

At this point we must discuss a suitable basis for expressing the amplitudes. We could use the basis (1.1), however this is not the most efficient option. By choosing a suitable basis of box functions we can considerably simplify the structure of the triangle coefficients.

Triangle integral functions may have one, two or three massless legs: $I_3^{1\mathrm{m}}$, $I_3^{2\mathrm{m}}$, $I_3^{3\mathrm{m}}$. The one-mass triangle function depends only on the momentum invariant of the massive

leg K_1 and is,

$$I_3^{1m} = \frac{r_\Gamma}{\epsilon^2}(-K_1^2)^{-1-\epsilon}\,, \tag{A.11}$$

while the two-mass triangle function with non-null momenta K_1 and K_2 is,

$$I_3^{2m} = \frac{r_\Gamma}{\epsilon^2}\frac{(-K_1^2)^{-\epsilon} - (-K_2^2)^{-\epsilon}}{(-K_1^2) - (-K_2^2)}\,. \tag{A.12}$$

Both of these integral functions contain $\ln(K^2)/\epsilon$ IR singularities. The key point is that the IR singularities of the amplitudes must be [38],

$$A_{IR}^{N=1 \text{ chiral}} = \frac{c_\Gamma}{\epsilon}A^{\text{tree}}\,, \qquad A_{IR}^{[0]} = \frac{c_\Gamma}{3\epsilon}A^{\text{tree}}\,, \tag{A.13}$$

so the $\ln(K^2)/\epsilon$ singularities must cancel. This constraint effectively determines the coefficients of I_3^{1m} and I_3^{2m} in terms of the box coefficients.

Specifically the one- and two-mass triangles are linear combinations of the set of functions,

$$G(-K^2) = r_\Gamma\frac{(-K^2)^{-\epsilon}}{\epsilon^2}\,, \tag{A.14}$$

with,

$$\begin{aligned}
I_3^{1m} &= G(-K_1^2)\,, \\
I_3^{2m} &= \frac{1}{(-K_1^2) - (-K_2^2)}\left(G(-K_1^2) - G(-K_2^2)\right).
\end{aligned} \tag{A.15}$$

The $G(-K^2)$ are labeled by the independent momentum invariants K^2 and in fact form an independent basis of functions, unlike the one and two-mass triangles which are not all independent.

In practice we need never calculate the coefficients of the G functions once we know the box coefficients. The only functions containing $\ln(s)/\epsilon$ terms are the box functions and I_3^{1m} and I_3^{2m} so,

$$\sum a_i I_4^i|_{\ln(K^2)/\epsilon} + b_G\frac{\ln(K^2)}{\epsilon} = 0\,. \tag{A.16}$$

This equation fixes the single b_G in terms of the a_i. The simplest approach to implement this simplification is to express the amplitudes in terms of truncated, finite $\mathcal{F}$-functions [3, 23, 39]:

$$\begin{aligned}
\mathcal{F}^{1m} &= F^{1m} + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon - \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon\,, \\
\mathcal{F}^{2me} &= F^{2me} + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon - \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_2^2}\right)^\epsilon - \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon\,, \\
\mathcal{F}^{2mh} &= F^{2mh} + \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon - \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_3^2}\right)^\epsilon - \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon\,, \\
\mathcal{F}^{3m} &= F^{3m} + \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-S}\right)^\epsilon + \frac{1}{\epsilon^2}\left(\frac{\mu^2}{-T}\right)^\epsilon - \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_2^2}\right)^\epsilon - \frac{1}{2\epsilon^2}\left(\frac{\mu^2}{-K_4^2}\right)^\epsilon\,.
\end{aligned} \tag{A.17}$$

The $\mathcal{N}=1$ and scalar amplitudes can then be expressed as,

$$A^{1-\text{loop}} = \sum a_i\mathcal{F}^i + \sum b_j^{3m}I_3^{3m,j} + \sum c_k I_2^k + R\,, \tag{A.18}$$

with no I_3^{1m} or I_3^{2m} present.

In transferring to this set of basis functions, the coefficients of the $\mathcal{F}^i$, $a_{\mathcal{F}}$, are simply related to the coefficients of the scalar box integrals, a_I, by,

$$a_{\mathcal{F}}^{1m} = -\frac{2}{ST}a_I^{1m}, \quad a_{\mathcal{F}}^{2me} = -\frac{2}{ST-K_2^2K_4^2}a_I^{2me},$$
$$a_{\mathcal{F}}^{2mh} = -\frac{2}{ST}a_I^{2mh}, \quad a_{\mathcal{F}}^{3m} = -\frac{2}{ST-K_2^2K_4^2}a_I^{3m}. \tag{A.19}$$

The remaining integral functions are the three-mass triangles I_3^{3m} as given, for example, in ref. [40] and the two-point bubble function,

$$I_2(P) = \frac{1}{\epsilon} + 2 - \ln(-P^2). \tag{A.20}$$

B Canonical forms for triple cuts

In this appendix the contributions of various canonical forms to the three-mass triangle coefficient are given. The triple cut is labelled as in figure 2.

$$\mathcal{E}_1 = \frac{\langle b\,\ell\rangle}{\langle a\,\ell\rangle} \longrightarrow E_1[a;b;\{K_j\}] = -\frac{\langle a|[K_1,K_3]|b\rangle}{2\langle a|K_1K_3|a\rangle}, \quad \langle a\,\hat{K}_i\rangle \neq 0,$$
$$\mathcal{J}_0 = \frac{\langle \ell\,a\rangle\langle \ell\,b\rangle}{\langle \ell|K_1K_2|\ell\rangle} \longrightarrow J_0[a,b;\{K_j\}] = \frac{\langle a|[K_1,K_2]|b\rangle}{\Delta_3},$$
$$\mathcal{J}_1^0 = \frac{\langle \ell\,a\rangle\langle \ell\,b\rangle\langle \ell\,c\rangle}{\langle \ell|K_1K_2|\ell\rangle\langle \ell\,d\rangle} \longrightarrow$$
$$J_1^0[d;a,b,c;\{K_j\}] = \frac{\langle b|[K_1,K_2]|d\rangle\langle c|[K_1,K_2]|a\rangle + \Delta_3\langle b\,d\rangle\langle c\,a\rangle}{2\Delta_3\langle d|K_1K_2|d\rangle} - \frac{\langle d\,b\rangle\langle d\,c\rangle\langle a|[K_1,K_2]|d\rangle}{2\langle d|K_1K_2|d\rangle^2}, \tag{B.1}$$

where,

$$\Delta_3 = 4(K_1\cdot K_2)^2 - 4K_1^2K_2^2 = (K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2(K_1^2K_2^2 + K_2^2K_3^2 + K_3^2K_1^2). \tag{B.2}$$

The above expressions are valid for $\ell = \ell_0$, $\ell = \ell_1$ and $\ell = \ell_2$ since,

$$\frac{\langle a|[K_1,K_3]|b\rangle}{2\langle a|K_1K_3|a\rangle} = \frac{\langle a|[K_1,K_2]|b\rangle}{2\langle a|K_1K_2|a\rangle} = \frac{\langle a|[K_2,K_3]|b\rangle}{2\langle a|K_2K_3|a\rangle}. \tag{B.3}$$

We can, as usual, extend these to, for example,

$$\mathcal{J}_n^0 = \frac{\langle \ell\,a\rangle\langle \ell\,b\rangle\prod_{i=1}^n\langle \ell\,c_i\rangle}{\langle \ell|K_1K_3|\ell\rangle\prod_{i=1}^n\langle \ell\,d_i\rangle} \longrightarrow J_n^0[\{d_i\};\{a,b,c_i\};\{K_j\}]. \tag{B.4}$$

The J_n^0 can be expressed in terms of J_1^0 as,

$$J_n^0[\{d_i\};\{a,b,c_i\};\{K_j\}] = \sum_{i=1}^n C_i J_1^0[d_i;a,b,c_n;\{K_j\}], \tag{B.5}$$

with,

$$C_i = \frac{\prod_{j=1}^{n-1}\langle c_j\,d_i\rangle}{\prod_{j=1,j\neq i}^n\langle d_j\,d_i\rangle}. \tag{B.6}$$

We also have terms which are of order ℓ^1:

$$\mathcal{E}_0^1 = [A|\ell_0|a\rangle \longrightarrow E_0^1[a; A; \{K_j\}] = -[A|a_0|a\rangle,$$

$$\mathcal{E}_1^1 = \frac{[A|\ell_0|b\rangle\langle\ell_0\, c\rangle}{\langle\ell_0\, d\rangle} \longrightarrow$$

$$E_1^1[d; b, c; A; \{K_j\}]$$

$$= -\frac{[A|a_0|b\rangle\langle d|[K_1, K_2]|c\rangle + [A|a_0|c\rangle\langle d|[K_1, K_2]|b\rangle - [A|a_0|d\rangle\langle b|[K_1, K_2]|c\rangle}{2\langle d|K_1 K_2|d\rangle}$$

$$+ \frac{\Delta_3[A|a_0|d\rangle\langle d\, b\rangle\langle d\, c\rangle}{2\langle d|K_1 K_2|d\rangle^2}, \tag{B.7}$$

$$\mathcal{J}_0^1 = \frac{[A|\ell_0|b\rangle\langle\ell_0\, c\rangle\langle\ell_0\, d\rangle}{\langle\ell_0|K_1 K_2|\ell_0\rangle} \longrightarrow$$

$$J_0^1[b, c, d; A; \{K_j\}]$$

$$= \frac{[A|a_0|b\rangle\langle c|[K_1, K_2]|d\rangle + [A|a_0|c\rangle\langle d|[K_1, K_2]|b\rangle + [A|a_0|d\rangle\langle b|[K_1, K_2]|c\rangle)\rangle}{8\Delta_3},$$

where,

$$a_0^\mu = \frac{-K_3^2(K_1^2 + K_2^2 - K_3^2)}{\Delta_3}K_1^\mu + \frac{K_1^2(K_3^2 + K_2^2 - K_1^2)}{\Delta_3}K_3^\mu. \tag{B.8}$$

Expressions for $\ell = \ell_1$ are obtained by replacing a_0 by a_1 where,

$$a_1^\mu = \frac{-K_1^2(K_2^2 + K_3^2 - K_2^2)}{\Delta_3}K_2^\mu + \frac{K_2^2(K_1^2 + K_3^2 - K_2^2)}{\Delta_3}K_1^\mu = a_0^\mu - K_1^\mu. \tag{B.9}$$

The $\ell = \ell_2$ expressions are obtained in a similar fashion.

Finally for order ℓ^1 we have,

$$\mathcal{J}_1^1 = \frac{[A|\ell_0|b\rangle\langle\ell_0\, c\rangle\langle\ell_0\, d\rangle\langle\ell_0\, e\rangle}{\langle\ell_0|K_1 K_2|\ell_0\rangle\langle\ell_0\, f\rangle} \longrightarrow$$

$$J_1^1[f; b, c, d, e; A; \{K_j\}] = \sum_{P_{12}} \frac{[A|a_0|b\rangle\langle f|[K_1, K_2]|c\rangle\langle d|[K_1, K_2]|e\rangle}{4\Delta_3\langle f|K_1 K_2|f\rangle}$$

$$- \sum_{P_6} \frac{[A|a_0|f\rangle\langle b|[K_1, K_2]|c\rangle\langle d|[K_1, K_2]|e\rangle}{24\Delta_3\langle f|K_1 K_2|f\rangle}$$

$$- \sum_{P_6} \frac{[A|a_0|f\rangle\langle f\, b\rangle\langle f\, c\rangle\langle d|[K_1, K_2]|e\rangle}{12\langle f|K_1 K_2|f\rangle^2} \tag{B.10}$$

$$- \sum_{P_4} \frac{[A|a_0|b\rangle\langle f|[K_1, K_2]|c\rangle\langle f|[K_1, K_2]|d\rangle\langle f|[K_1, K_2]|e\rangle}{4\Delta_3\langle f|K_1 K_2|f\rangle^2}$$

$$- \sum_{P_6} \frac{[A|a_0|f\rangle\langle f\, b\rangle\langle f\, c\rangle\langle f|[K_1, K_2]|d\rangle\langle f|[K_1, K_2]|e\rangle}{12\langle f|K_1 K_2|f\rangle^3}.$$

For QCD amplitude we generically need forms of order ℓ^2,

$$\mathcal{E}_0^2 = [A|\ell_0|a\rangle[B|\ell_0|b\rangle \longrightarrow$$

$$E_0^2[a, b; A, B; \{K_j\}] = -\frac{1}{2}\sum_{P_2}[A|a_0|a\rangle[B|a_0|b\rangle - \frac{K_1^2 K_2^2 K_3^2}{2\Delta_3^2}[A|[K_1, K_2]|B]\langle a|[K_1, K_2]|b\rangle,$$

$$\mathcal{E}_1^2 = \frac{[A|\ell_0|a\rangle[B|\ell_0|b\rangle\langle\ell_0\,c\rangle}{\langle\ell_0\,d\rangle} \longrightarrow$$

$$E_1^2[d; a, b, c; A, B; \{K_j\}] = -\frac{[A|a_0|d\rangle[B|a_0|d\rangle\langle d|[K_1, K_2]|a\rangle\langle d|[K_1, K_2]|b\rangle\langle d|[K_1, K_2]|c\rangle}{2\langle d|K_1 K_2|d\rangle^3}$$

$$+ \sum_{P_6} \frac{[A|a_0|d\rangle[B|a_0|a\rangle\langle d|[K_1, K_2]|b\rangle\langle d|[K_1, K_2]|c\rangle}{3\langle d|K_1 K_2|d\rangle^2}$$

$$+ [A|a_0|d\rangle[B|a_0|d\rangle \sum_{P_3} \frac{\langle d|[K_1, K_2]|a\rangle\langle b|[K_1, K_2]|c\rangle}{6\langle d|K_1 K_2|d\rangle^2}$$

$$- \sum_{P_6} \frac{[A|a_0|d\rangle[B|a_0|a\rangle\langle b|[K_1, K_2]|c\rangle}{4\langle d|K_1 K_2|d\rangle}$$

$$- \sum_{P_6} \frac{[A|a_0|a\rangle[B|a_0|b\rangle\langle d|[K_1, K_2]|c\rangle}{3\langle d|K_1 K_2|d\rangle}$$

$$- \sum_{P_3} \frac{K_1^2 K_2^2 K_3^2 [A|[K_1, K_2]|B\rangle\langle d|[K_1, K_2]|a\rangle\langle b|[K_1, K_2]|c\rangle}{12\Delta_3^2\langle d|K_1 K_2|d\rangle} \,,$$

$$\mathcal{J}_0^2 = \frac{[A|\ell_0|a\rangle[B|\ell_0|b\rangle\langle\ell_0\,c\rangle\langle\ell_0\,d\rangle}{\langle\ell_0|K_1 K_2|\ell_0\rangle} \longrightarrow$$

$$J_0^2[a, b, c, d; A, B; \{K_j\}] = \frac{1}{6\Delta_3} \sum_{P_{12}} [A|a_0|a\rangle[B|a_0|b\rangle\langle c|[K_1, K_2]|d\rangle$$

$$+ \frac{K_1^2 K_2^2 K_3^2}{6\Delta_3^3} \sum_{P_3} [A|[K_1, K_2]|B\rangle\langle a|[K_1, K_2]|b\rangle\langle c|[K_1, K_2]|d\rangle \,,$$

$$\text{(B.11)}$$

$$\mathcal{J}_1^2 = \frac{[A|\ell_0|a\rangle[B|\ell_0|b\rangle\langle\ell_0\,c\rangle\langle\ell_0\,d\rangle\langle\ell_0\,e\rangle}{\langle\ell_0|K_1 K_2|\ell_0\rangle\langle\ell_0\,f\rangle} \longrightarrow$$

$$J_1^2[f; a, b, c, d, e; A, B; \{K_j\}]$$

$$= \frac{K_1^2 K_2^2 K_3^2 [A|[K_1, K_2]|B\rangle}{20\langle f|K_1 K_2|f\rangle\Delta_3^3} \sum_{P_{15}} \langle f|[K_1, K_2]|a\rangle\langle b|[K_1, K_2]|c\rangle\langle d|[K_1, K_2]|e\rangle$$

$$- \frac{1}{60\langle f|K_1 K_2|f\rangle\Delta_3} \sum_{P_{30}} [A|a_0|f\rangle[B|a_0|a\rangle\langle b|[K_1, K_2]|c\rangle\langle d|[K_1, K_2]|e\rangle$$

$$+ \frac{1}{40\langle f|K_1 K_2|f\rangle\Delta_3} \sum_{P_{60}} [A|a_0|a\rangle[B|a_0|b\rangle\langle c|[K_1, K_2]|d\rangle\langle f|[K_1, K_2]|e\rangle$$

$$- \frac{K_1^2 K_2^2 K_3^2 [A|[K_1, K_2]|B\rangle}{40\langle f|K_1 K_2|f\rangle^2\Delta_3^3} \sum_{P_{10}} \langle a|[K_1, K_2]|b\rangle\langle f|[K_1, K_2]|c\rangle\langle f|[K_1, K_2]|d\rangle\langle f|[K_1, K_2]|e\rangle$$

$$- \frac{1}{40\langle f|K_1 K_2|f\rangle^2} \sum_{P_{60}} [A|a_0|a\rangle[B|a_0|f\rangle\langle b|[K_1, K_2]|c\rangle\langle f\,d\rangle\langle f\,e\rangle$$

$$- \frac{1}{30\langle f|K_1 K_2|f\rangle^2} \sum_{P_{60}} [A|a_0|a\rangle[B|a_0|b\rangle\langle f|[K_1, K_2]|c\rangle\langle f\,d\rangle\langle f\,e\rangle$$

$$+ \frac{1}{30\langle f|K_1 K_2|f\rangle^3} \sum_{P_{60}} [A|a_0|a\rangle[B|a_0|f\rangle\langle f|[K_1, K_2]|b\rangle\langle f|[K_1, K_2]|c\rangle\langle f\,d\rangle\langle f\,e\rangle$$

$$- \frac{[A|a_0|f\rangle[B|a_0|f\rangle}{60\langle f|K_1 K_2|f\rangle^3} \sum_{P_{30}} \langle a|[K_1, K_2]|b\rangle \langle f|[K_1, K_2]|c\rangle \langle f\, d\rangle \langle f\, e\rangle$$

$$- \frac{[A|a_0|f\rangle[B|a_0|f\rangle}{20\langle f|K_1 K_2|f\rangle^4} \sum_{P_{10}} \langle f|[K_1, K_2]|a\rangle \langle f|[K_1, K_2]|b\rangle \langle f|[K_1, K_2]|c\rangle \langle f\, d\rangle \langle f\, e\rangle\,. \qquad \text{(B.12)}$$

Cubic and higher order terms can also be evaluated. For example,

$$[A|\ell_0|a\rangle[B|\ell_0|b\rangle[C|\ell_0|c\rangle \longrightarrow -\frac{1}{6} \sum_{P_6} [A|a_0|a\rangle[B|a_0|b\rangle[C|a_0|c\rangle$$
$$- \frac{K_1^2 K_2^2 K_3^2}{6\Delta_3^2} \sum_{P_9} [A|[K_1, K_2]|B\rangle \langle a|[K_1, K_2]|b\rangle [C|a_0|c\rangle\,, \qquad \text{(B.13)}$$

however we can, in general, avoid these in QCD calculations.

C   All-n expressions for coefficients

C.1   Box coefficients

Many of the box coefficients that appear in the seven-point $\mathcal{N} = 1$ amplitudes are special cases of general n-point expressions. We gather these together here and give their specialisations to the box coefficients of section 4. We denote the external legs contributing to the non-null momentum K_i by $\{f_i, \ldots, u_i\}$.

The three-mass boxes in a NMHV $\mathcal{N} = 1$ amplitude vanish unless exactly one negative helicity gluon is attached to each non-null vertex. We denote this single negative helicity gluon on vertex i by m_{i-1}.
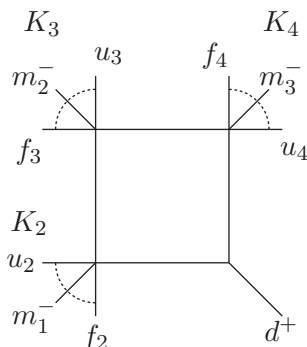

$$c^{3m}[m_1, m_2, m_3; d, K_2, K_3, K_3] = -\frac{\mathcal{H}^2 \langle u_2\, f_3\rangle \langle u_3\, f_4\rangle \langle m_3\, d\rangle \langle m_1\, d\rangle}{2\langle 1\, 2\rangle \langle 2\, 3\rangle \ldots \langle n\, 1\rangle}$$
$$\times \frac{\langle m_1|K_3 K_4|d\rangle \langle m_3|K_3 K_2|d\rangle \langle m_2|K_3 K_2|d\rangle \langle m_2|K_3 K_4|d\rangle [d|K_2 K_3 K_4|d\rangle}{\langle d|K_2 K_3|d\rangle^2 \langle d|K_4 K_3|u_2\rangle \langle d|K_4 K_3|f_3\rangle \langle d|K_2 K_3|u_3\rangle \langle d|K_2 K_3|f_4\rangle K_3^2} \tag{C.1}$$

where,

$$\mathcal{H} = \langle m_1\, m_2\rangle \langle d|K_2 K_3|m_3\rangle + \langle m_3\, m_2\rangle \langle d|K_4 K_3|m_1\rangle. \tag{C.2}$$

These box coefficients are close in form to those of $\mathcal{N} = 4$ Yang-Mills [13, 15].

All of the NMHV two mass boxes we need can be found using Generalised Unitarity taking MHV and $\overline{\text{MHV}}$ tree amplitudes as the inputs.
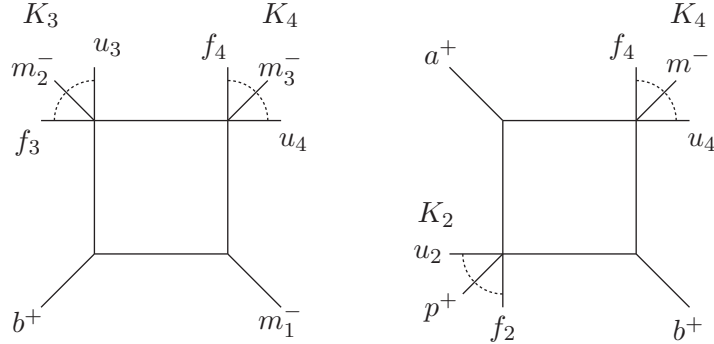
With the labellings given in the figure, the coefficients are,

$$
\begin{aligned}
c^{2mh} =\; & \frac{P^2[m_1\,b]^2\langle m_3|P|m_1]\langle m_2|P|m_1]\langle m_2\,b\rangle\langle m_3|K_4P|b\rangle}{2K_4^2[m_1|K_3P|m_1]\langle b|PK_4|b\rangle\langle f_4|P|m_1]\langle u_3|P|m_1]\langle b|PK_4|u_4\rangle\langle f_3\,b\rangle[m_1|P|b\rangle^2} \\
& \times \frac{(P^2\langle b|m_1P|b\rangle\langle m_2\,m_3\rangle + \langle b|PK_4|b\rangle\langle m_3|m_1P|m_2\rangle)^2}{\prod_{i=f_3}^{u_3-1}\langle i\,i+1\rangle\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle}, \\
c^{2me} =\; & \frac{[p|K_2|m\rangle^2[b|K_2|a\rangle[a|K_2|b\rangle[p|K_2|a\rangle[p|K_2|b\rangle\langle m\,a\rangle\langle m\,b\rangle}{2\langle a\,f_4\rangle\langle u_4\,b\rangle\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle\prod_{i=f_2}^{u_2-1}[i\,i+1]\langle a\,b\rangle^2[f_2|K_2|a\rangle[u_2|K_2|b\rangle K_2^2},
\end{aligned}
\tag{C.3}
$$

where, $P = K_3 + k_b = -(K_4 + k_{m_1})$.

One mass boxes in $\mathcal{N} = 1$ amplitudes have either one or two negative helicity external legs attached to the three-point corners. In the latter case, the third negative helicity leg of any NMHV amplitude attaches to an MHV corner and a general form exists:
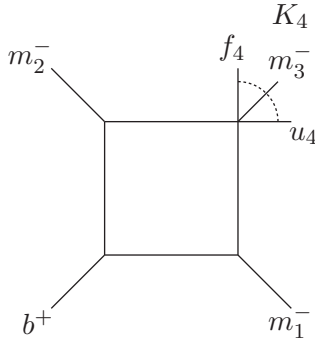
$$
c_0^{1m} = -\frac{[b|K_4|m_3\rangle^2[m_1|K_4|m_3\rangle[m_2|K_1|m_3\rangle s_{m_1b}s_{bm_2}}{2K_4^2[m_1\,m_2]^2[m_1|K_4|m_2+1\rangle[m_2|K_4|u_4\rangle\prod_{i=f_4}^{u_4-1}\langle i\,i+1\rangle}.
\tag{C.4}
$$

When there are two negative helicity external legs attached to the massive corner, NMHV tree amplitudes are required and we must consider each helicity configuration separately

as illustrated.
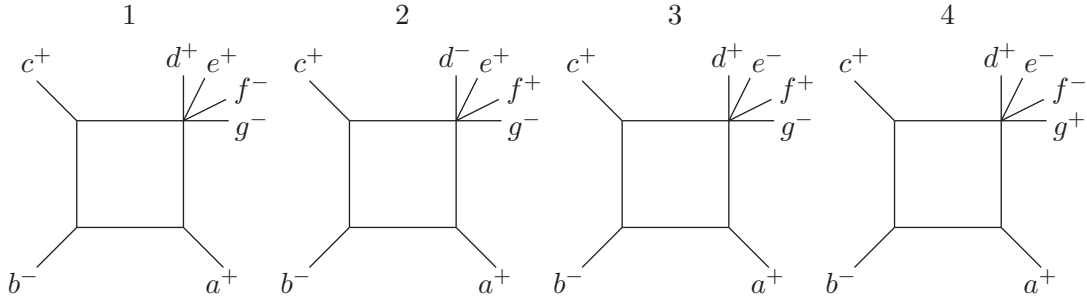
$$c_1^{1m} = \frac{\langle b|P_{abc}P_{def}|f\rangle^2\langle c|P_{abc}P_{def}|f\rangle s_{ab}s_{bc}}{2t_{def}t_{abc}\langle a\,c\rangle^2\langle d\,e\rangle\langle e\,f\rangle[g|P_{ab}|c\rangle[g|P_{def}|d\rangle} - \frac{[e|P_{efg}|b\rangle^2[e|P_{efg}|c\rangle s_{ab}s_{bc}}{2\langle a\,c\rangle^2[e\,f][f\,g]\langle c\,d\rangle[g|P_{efg}|d\rangle t_{efg}},$$

$$c_2^{1m} = \frac{s_{ab}s_{bc}\langle b\,g\rangle^2\langle c\,g\rangle[e\,f]^3}{2t_{def}\langle a\,c\rangle^2[d\,e][d|P_{def}|g\rangle[f|P_{def}|c\rangle}$$

$$+ \frac{s_{ab}s_{bc}([f\,g]\langle g\,d\rangle\langle b\,c\rangle - [f|P_{abc}|c\rangle\langle d\,b\rangle)^2([f\,g]\langle g\,d\rangle\langle a\,c\rangle - [f|P_{abc}|c\rangle\langle d\,a\rangle)[f|P_{abc}|c\rangle}{2\langle a\,c\rangle^2[f\,g][g|P_{abc}|c\rangle\langle d\,e\rangle[f|P_{de}|c\rangle\langle c|P_{abc}P_{fg}|e\rangle(s_{de}\langle c\,a\rangle + \langle c|P_{de}P_{bc}|a\rangle)}$$

$$- \frac{s_{ab}s_{bc}\langle b|P_{abc}P_{efg}|g\rangle^2\langle a|P_{abc}P_{efg}|g\rangle\langle c|P_{abc}P_{efg}|g\rangle}{2t_{efg}t_{abc}\langle a\,c\rangle^2[d|P_{efg}|g\rangle[d|P_{abc}|a\rangle\langle e\,f\rangle\langle f\,g\rangle\langle c|P_{abc}P_{efg}|e\rangle}, \tag{C.5}$$

$$c_3^{1m} = \frac{s_{ab}s_{bc}\langle b|P_{abc}P_{def}|e\rangle^2\langle c|P_{abc}P_{def}|e\rangle\langle a|P_{abc}P_{def}|e\rangle}{2t_{def}t_{abc}\langle c\,a\rangle^2\langle d\,e\rangle\langle e\,f\rangle[g|P_{abc}|c\rangle[g|P_{def}|d\rangle\langle a|P_{abc}P_{def}|f\rangle}$$

$$- \frac{s_{ab}s_{bc}[f|P_{efg}|b\rangle^2[f|P_{efg}|a\rangle[f|P_{efg}|c\rangle}{2t_{efg}\langle a\,c\rangle^2\langle c\,d\rangle[e\,f][f\,g][e|P_{efg}|a\rangle[g|P_{efg}|d\rangle}$$

$$- \frac{s_{ab}s_{bc}([d|P_{abc}|a\rangle\langle g\,b\rangle + [d\,e]\langle e\,g\rangle\langle a\,b\rangle)^2([d|P_{abc}|a\rangle\langle g\,c\rangle + [d\,e]\langle e\,g\rangle\langle a\,c\rangle)}{2\langle c\,a\rangle^2\langle f\,g\rangle[d\,e][e|P_{efg}|a\rangle\langle|P_{abc}P_{def}|f\rangle(s_{de}\langle c\,a\rangle + \langle c|P_{de}P_{abc}|a\rangle)}, $$

$$c_4^{1m} = - \frac{s_{ab}s_{bc}\langle f\,a\rangle(\langle f\,a\rangle[d|P_{abc}|b\rangle - \langle a\,b\rangle[d\,g]\langle g\,f\rangle)^2(\langle f\,a\rangle[d|P_{abc}|c\rangle - \langle a\,c\rangle[d\,g]\langle g\,f\rangle)}{2\langle c\,a\rangle^2\langle f\,g\rangle\langle g\,a\rangle[d\,e]\langle a|P_{abc}P_{def}|f\rangle[e|P_{efg}|a\rangle(s_{de}\langle c\,a\rangle + \langle c|P_{de}P_{abc}|a\rangle)}$$

$$+ \frac{s_{ab}s_{bc}[g|P_{abc}|b\rangle^2[g|P_{abc}|a\rangle\langle e\,f\rangle^3}{2\langle a\,c\rangle^2 t_{def}t_{abc}\langle d\,e\rangle[g|P_{def}|d\rangle\langle a|P_{abc}P_{def}|f\rangle}$$

$$- \frac{s_{ab}s_{bc}[g|P_{efg}|b\rangle^2[g|P_{efg}|a\rangle[g|P_{efg}|c\rangle}{2t_{efg}\langle a\,c\rangle^2\langle c\,d\rangle[e\,f][f\,g][e|P_{efg}|a\rangle[g|P_{efg}|d\rangle}. \tag{C.6}$$

C.2 Particular bubble coefficients

Our first example is the bubble coefficient in an $\mathcal{N} = 1$ MHV amplitude. Coefficients of $\ln(-P_{a\cdots b}^2)$ vanish unless exactly one of the two negative helicity legs lies within $P_{a\cdots b}$. The non-vanishing coefficient of $\ln(-P_{a\cdots b}^2)$ is then,

$$\frac{\langle m_1\,m_2\rangle^2\langle b\,b+1\rangle\langle a-1\,a\rangle}{\prod_i\langle i\,i+1\rangle} \times H_4[a-1,a,b,b+1;m_1,m_1,m_2,m_2;P_{a\cdots b}]. \tag{C.7}$$

The second example is relevant for the seven-point NMHV amplitudes. Consider the case where one side of the cut is MHV and the other is $\overline{\text{MHV}}$. Let m be the single negative helicity on the MHV side which contains legs $a\cdots b$ and p be the single positive helicity on

the $\overline{\text{MHV}}$ side. The coefficient of $\ln(-P_{a\cdots b}^2)$ is then just,

$$\frac{[p|P_{a\cdots b}|m\rangle^2}{\prod_{i=a}^{b-1}\langle i\,i+1\rangle \prod_{j=b+1}^{a-2}[j\,j+1]P_{a\cdots b}^2} \tag{C.8}$$
$$\times H_4[a, b, P_{a\cdots b}|a-1], P_{a\cdots b}|b+1]; m, m, P_{a\cdots b}|p], P_{a\cdots b}|p]; P_{a\cdots b}]\,.$$

Since many of the d_i coefficients in the seven-point NMHV amplitudes are special cases of this generic expression specialised to seven-point we define,

$$C_0[a, b, c, d, e, f, g; p, m]$$
$$\equiv \frac{[p|P_{gab}|m\rangle^2}{\langle c\,d\rangle\langle d\,e\rangle\langle e\,f\rangle[g\,a][a\,b]P_{gab}^2} \times H_4[c, f, P_{gab}|b], P_{gab}|g]; m, m, P_{gab}|p], P_{gab}|p]; P_{a\cdots b}]\,. \tag{C.9}$$

D Seven-point tree expressions

We have:
$$A : A(s_1, \bar{s}_2, 3^-, 4^-, 5^+, 6^+, 7^+) = T_{1a}^A + T_{1b}^A + T_2^A + T_3^A\,, \tag{D.1}$$

with,

$$T_{1a}^A = \frac{[5|P_{345}|2\rangle[5|P_{345}|1\rangle^2}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 1\,2\rangle[3\,4][4\,5][3|P_{345}|6\rangle t_{345}} \times \left(\frac{[5|P_{345}|1\rangle}{[5|P_{345}|2\rangle}\right)^{2h}\,,$$

$$T_{1b}^A = \frac{\langle 6|P_{71}P_{23}|4\rangle([2|P_{67}|1\rangle\langle 6\,4\rangle - [2|5|4\rangle\langle 1\,6\rangle)^2}{\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,1\rangle[2\,3][3|P_{712}|6\rangle[2|P_{71}|6\rangle\langle 1|P_{23}P_{45}|6\rangle}$$
$$\times \left(\frac{([2|P_{67}|1\rangle\langle 6\,4\rangle - [2|5|4\rangle\langle 1\,6\rangle)}{\langle 6|P_{71}P_{23}|4\rangle}\right)^{2h}\,, \quad \text{(D.2)}$$

$$T_2^A = \frac{\langle 3\,4\rangle^4}{\langle 3\,5\rangle^4} \times T_2^B\,,$$

$$T_3^A = \frac{[7|P_{456}|4\rangle^4}{[7|P_{456}|5\rangle^4} \times T_3^B\,.$$

$$B : A(s_1, \bar{s}_2, 3^-, 4^+, 5^-, 6^+, 7^+) = T_{1a}^B + T_{1b}^B + T_{1c}^B + T_2^B + T_3^B + T_4^B\,, \tag{D.3}$$

with,

$$T_{1a}^B = \frac{[4|P_{234}|5\rangle^2[2\,4]^2\langle 5\,1\rangle^2}{\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,1\rangle[2\,3][3\,4]t_{234}[2|P_{234}|5\rangle[4|P_{234}|1\rangle} \times \left(\frac{[2\,4]\langle 5\,1\rangle}{[4|P_{234}|5\rangle}\right)^{2h}\,,$$

$$T_{1b}^B = \frac{-\langle 2\,3\rangle\langle 1\,3\rangle^2[4|P_{67}|1\rangle^4}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 1\,2\rangle[4\,5][5|P_{67}|1\rangle[4|P_{23}|1\rangle\langle 1|P_{67}P_{45}|3\rangle\langle 6|P_{45}P_{23}|1\rangle} \times \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h}\,,$$

$$T_{1c}^B = \frac{t_{671}[2|P_{671}|1\rangle^2\langle 3\,5\rangle^4}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 3\,4\rangle\langle 4\,5\rangle[2|P_{71}|6\rangle[2|P_{345}|5\rangle\langle 1|P_{67}P_{45}|3\rangle t_{345}} \times \left(\frac{-[2|P_{671}|1\rangle}{t_{671}}\right)^{2h}\,,$$

$$T_2^B = \frac{-[2\,7]^2[1\,7]\langle 3\,5\rangle^4}{[1\,2]\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle[2|P_{712}|6\rangle[7|P_{712}|3\rangle t_{712}} \times \left(-\frac{[2\,7]}{[1\,7]}\right)^{2h}\,,$$

$$T_3^B = \frac{\langle 1\,3\rangle^2\langle 2\,3\rangle^2[7|P_{123}|5\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 4\,5\rangle\langle 5\,6\rangle[7|P_{123}|3\rangle[7|P_{123}|4\rangle\langle 1|P_{23}P_{45}|6\rangle t_{123}t_{456}} \times \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h}\,,$$

$$T_4^B = \frac{-\langle 1\,3\rangle^2 \langle 2\,3\rangle^2 \langle 5\,6\rangle^2 [6\,7]^3}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 5\,6\rangle [7|P_{56}|4\rangle [5|P_{567}|1\rangle t_{567} s_{56}} \times \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h}. \tag{D.4}$$

$$C : A(s_1, \bar{s}_2, 3^-, 4^+, 5^+, 6^-, 7^+) = T_{1a}^C + T_{1b}^C + T_{1c}^C + T_2^C + T_3^C + T_4^C, \tag{D.5}$$

with,

$$T_{1a}^C = \frac{[2\,4]^2 [4|P_{234}|6\rangle^2 \langle 1\,6\rangle^2}{[2\,3][3\,4]\langle 5\,6\rangle \langle 6\,7\rangle \langle 7\,1\rangle [4|P_{234}|1\rangle [2|P_{234}|5\rangle t_{234}} \times \left(\frac{-[2\,4]\langle 1\,6\rangle}{[4|P_{234}|6\rangle}\right)^{2h},$$

$$T_{1b}^C = -\frac{\langle 1\,3\rangle^2 \langle 2\,3\rangle [4\,5]^3 \langle 1\,6\rangle^4}{\langle 6\,7\rangle \langle 7\,1\rangle \langle 1\,2\rangle [4|P_{234}|1\rangle [5|P_{234}|1\rangle \langle 1|P_{67}P_{45}|3\rangle \langle 1|P_{23}P_{45}|6\rangle} \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h},$$

$$T_{1c}^C = \frac{[2|P_{345}|3\rangle^2 [6|P_{71}P_{345}|3\rangle^2 \langle 1\,6\rangle^2}{\langle 6\,7\rangle \langle 7\,1\rangle \langle 3\,4\rangle \langle 4\,5\rangle [2|P_{71}|6\rangle [2|P_{345}|5\rangle \langle 1|P_{671}P_{345}|3\rangle t_{345} t_{671}} \times \left(\frac{[2|P_{345}|3\rangle \langle 1\,6\rangle}{\langle 6|P_{671}P_{345}|3\rangle}\right)^{2h},$$

$$T_2^C = \frac{[1\,7]^2 [2\,7]^2 \langle 3\,6\rangle^4}{[7\,1][1\,2]\langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,6\rangle [7|P_{712}|3\rangle [2|P_{712}|6\rangle t_{712}} \left(-\frac{[2\,7]}{[1\,7]}\right)^{2h},$$

$$T_3^C = \frac{\langle 1\,3\rangle^2 \langle 2\,3\rangle^2 [7|P_{456}|6\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 4\,5\rangle \langle 5\,6\rangle \langle 1|P_{123}P_{456}|6\rangle [7|P_{456}|4\rangle [7|P_{456}|3\rangle t_{123} t_{456}} \left(\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\right)^{2h},$$

$$T_4^C = \frac{[7\,5]^4 \langle 1\,3\rangle^2 \langle 2\,3\rangle^2}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle [5\,6][6\,7][5|P_{567}|1\rangle [7|P_{567}|4\rangle t_{567}} \left(\frac{\langle 3\,1\rangle}{\langle 3\,2\rangle}\right)^{2h}. \tag{D.6}$$

$$D : A_7(s_1, \bar{s}_2, 3^-, 4^+, 5^+, 6^+, 7^-) = T_1^D + T_2^D + T_{4a}^D + T_{4b}^D + T_{4c}^D, \tag{D.7}$$

with,

$$T_1^D = -\frac{[6|P_{71}|3\rangle^2 \langle 3\,2\rangle [1\,6]^2}{[6\,7][7\,1]\langle 3\,4\rangle \langle 4\,5\rangle [1|P_{671}|5\rangle [6|P_{671}|2\rangle t_{671}} \times \left(\frac{[6|P_{71}|3\rangle}{[1\,6]\langle 3\,2\rangle}\right)^{2h},$$

$$T_2^D = -\frac{\langle 7\,1\rangle \langle 7\,2\rangle^2 [6|P_{712}|3\rangle^3}{\langle 1\,2\rangle \langle 3\,4\rangle \langle 4\,5\rangle [6|P_{712}|2\rangle \langle 5|P_{34}P_{12}|7\rangle t_{345} t_{712}} \times \left(\frac{\langle 7\,1\rangle}{\langle 7\,2\rangle}\right)^{2h},$$

$$T_{4a}^D = \frac{[4|P_{56}|7\rangle^3 \langle 7\,2\rangle^2 \langle 7\,1\rangle^2}{\langle 5\,6\rangle \langle 6\,7\rangle \langle 7\,1\rangle \langle 1\,2\rangle [3\,4][3|P_{12}|7\rangle \langle 7|P_{56}P_{34}|2\rangle \langle 7|P_{12}P_{34}|5\rangle} \times \left(\frac{\langle 7\,1\rangle}{\langle 7\,2\rangle}\right)^{2h},$$

$$T_{4b}^D = \frac{\langle 2\,3\rangle [1|P_{56}|7\rangle^2 \langle 7|P_{56}P_{234}|3\rangle^2}{\langle 5\,6\rangle \langle 6\,7\rangle \langle 3\,4\rangle [1|P_{567}|5\rangle [1|P_{234}|4\rangle \langle 7|P_{56}P_{234}|2\rangle t_{234} t_{567}} \times \left(\frac{-\langle 7|P_{56}P_{234}|3\rangle}{\langle 2\,3\rangle [1|P_{56}|7\rangle}\right)^{2h},$$

$$T_{4c}^D = \frac{[1|P_{123}|7\rangle^2 [2|P_{123}|7\rangle^2}{\langle 4\,5\rangle \langle 5\,6\rangle \langle 6\,7\rangle [1\,2][2\,3][3|P_{123}|7\rangle [1|P_{123}|4\rangle t_{123}} \times \left(\frac{-[2|P_{123}|7\rangle}{[1|P_{123}|7\rangle}\right)^{2h}. \tag{D.8}$$

$$E : A_7(s_1, \bar{s}_2, 3^+, 4^-, 5^-, 6^+, 7^+) = T_{1a}^E + T_{1b}^E + T_{1c}^E + T_2^E + T_3^E + T_4^E, \tag{D.9}$$

with,

$$T_{1a}^E = \frac{\langle 2\,4\rangle^2 t_{671}\langle 1|P_{67}P_{23}|4\rangle^2}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 2\,3\rangle\langle 3\,4\rangle[5|P_{671}|1\rangle[5|P_{671}|2\rangle\langle 6|P_{71}P_{23}|4\rangle t_{234}}\left(\frac{-\langle 1|P_{67}P_{23}|4\rangle}{\langle 2\,4\rangle t_{671}}\right)^{2h},$$

$$T_{1b}^E = \frac{-[3|P_{71}|6\rangle^2[2\,3]\langle 4\,5\rangle^3\langle 1\,6\rangle^2}{\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,1\rangle[2|P_{71}|6\rangle[3|P_{45}|6\rangle\langle 6|P_{71}P_{23}|4\rangle\langle 1|P_{23}P_{45}|6\rangle}\left(\frac{-[2\,3]\langle 6\,1\rangle}{[3|P_{71}|6\rangle}\right)^{2h},$$

$$T_{1c}^E = \frac{[3|P_{345}|2\rangle^2[3|P_{345}|1\rangle^2}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 1\,2\rangle[3\,4][4\,5][5|P_{345}|2\rangle[3|P_{345}|6\rangle t_{345}}\left(\frac{[3|P_{345}|1\rangle}{[3|P_{345}|2\rangle}\right)^{2h},$$

$$T_2^E = \left(\frac{\langle 4\,5\rangle}{\langle 3\,5\rangle}\right)^4 T_2^B,$$

$$T_3^E = \frac{[7|P_{456}|1\rangle^2[7|P_{456}|2\rangle^2\langle 4\,5\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 5\,6\rangle[7|P_{456}|3\rangle[7|P_{56}|4\rangle\langle 1|P_{23}P_{45}|6\rangle t_{123}t_{456}}\times\left(\frac{[7|P_{456}|1\rangle}{[7|P_{456}|2\rangle}\right)^{2h},$$

$$T_4^E = \left(\frac{\langle 1\,4\rangle^2\langle 2\,4\rangle^2}{\langle 1\,3\rangle^2\langle 2\,3\rangle^2}\right)\left(\frac{\langle 2\,3\rangle\langle 1\,4\rangle}{\langle 1\,3\rangle\langle 2\,4\rangle}\right)^{2h}T_4^B.$$

$$\text{(D.10)}$$

$$F: A_7(s_1,\bar{s}_2,3^+,4^-,5^+,6^-,7^+) = T_{1a}^F + T_{1b}^F + T_{1c}^F + T_2^F + T_3^F + T_4^F, \qquad \text{(D.11)}$$

with,

$$T_{1a}^F = \frac{[5|P_{234}|4\rangle^2\langle 2\,4\rangle^2[5|P_{71}|6\rangle^2\langle 1\,6\rangle^2}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 2\,3\rangle\langle 3\,4\rangle[5|P_{234}|1\rangle[5|P_{234}|2\rangle\langle 6|P_{71}P_{23}|4\rangle t_{671}t_{234}}\left(\frac{[5|P_{234}|4\rangle\langle 1\,6\rangle}{\langle 2\,4\rangle[5|P_{234}|6\rangle}\right)^{2h},$$

$$T_{1b}^F = \frac{-[2\,3]^2[3|P_{71}|6\rangle^2\langle 4\,6\rangle^4\langle 1\,6\rangle^2}{\langle 6\,7\rangle\langle 7\,1\rangle[2|P_{71}|6\rangle[2\,3]\langle 4\,5\rangle\langle 5\,6\rangle[3|P_{45}|6\rangle\langle 6|P_{71}P_{23}|4\rangle\langle 1|P_{23}P_{45}|6\rangle}\left(\frac{-[2\,3]\langle 6\,1\rangle}{[3|P_{71}|6\rangle}\right)^{2h},$$

$$T_{1c}^F = \frac{\langle 6\,2\rangle^2\langle 1\,6\rangle^2[3\,5]^4}{\langle 6\,7\rangle\langle 7\,1\rangle\langle 1\,2\rangle[3\,4][4\,5][5|P_{345}|2\rangle[3|P_{345}|6\rangle t_{345}}\left(\frac{\langle 6\,1\rangle}{\langle 6\,2\rangle}\right)^{2h},$$

$$T_2^F = \frac{\langle 4\,6\rangle^4}{\langle 3\,6\rangle^4}T_2^C,$$

$$T_3^F = \frac{\langle 4\,6\rangle^4[7|P_{123}|1\rangle^2[7|P_{123}|2\rangle^2}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 4\,5\rangle\langle 5\,6\rangle[7|P_{123}|3\rangle[7|P_{123}|4\rangle\langle 1|P_{23}P_{45}|6\rangle t_{123}t_{456}}\times\left(\frac{[7|P_{123}|1\rangle}{[7|P_{123}|2\rangle}\right)^{2h},$$

$$T_4^F = \frac{\langle 1\,4\rangle^{2+2h}\langle 2\,4\rangle^{2-2h}}{\langle 1\,3\rangle^{2+2h}\langle 2\,3\rangle^{2-2h}}T_4^C.$$

$$\text{(D.12)}$$

E D_X functions

The function D_A was given in section 4.5. The remaining five D_X functions are given by:

$$D_A(a,b,c,d,e,f,g) =$$

$$-\frac{[e|P_{cde}|a\rangle^2}{\langle a\,b\rangle\langle f\,g\rangle[c\,d][d\,e][c|P_{cde}|f\rangle t_{cde}}H_2[b,g;a,P_{cde}|e];P_{ab}]$$

$$+\frac{\langle f\,d\rangle}{[a\,b]\langle d\,e\rangle\langle e\,f\rangle\langle f\,g\rangle[c|P_{de}|f\rangle}\overline{H_4}[a,c,P_{cde}|f\rangle,P_{ab}P_{abc}P_{de}|f\rangle;b,g,XA,XA;P_{ab}]$$

$$+\frac{1}{[a\,b]\langle d\,e\rangle\langle e\,f\rangle\langle f\,g\rangle[c|P_{de}|f\rangle}$$

$$\times\overline{H_5}[a,P_{ab}|g\rangle,c,P_{cde}|f\rangle,P_{ab}P_{abc}P_{de}|f\rangle;b,P_{ab}|f\rangle,P_{ef}|d\rangle,XA,XA;P_{ab}]$$

$$+ \frac{\langle c\,d\rangle^4[g\,b]^2}{[a\,b]\langle c\,d\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{ab}|c\rangle t_{gab}} H_2[b, P_{ab}P_{gab}|f\rangle; a, P_{ab}|g], P_{ab}]$$

$$+ \frac{\langle a\,c\rangle^2[g|P_{abc}|d\rangle^4}{\langle a\,b\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{ab}|c\rangle[g|P_{abc}|d\rangle t_{abc}t_{def}} H_2[b, P_{abc}P_{de}|f\rangle; a, c, P_{ab}], \qquad (E.1)$$

where,

$$\bar\lambda_{XA}(a,b,c,d,e,f,g) = \bar\lambda_g\langle g\,a\rangle[a\,b]\langle f\,d\rangle + \bar\lambda_f\langle f\,a\rangle[a\,b]\langle f\,d\rangle + \bar\lambda_e\langle e\,d\rangle[a\,b]\langle f\,a\rangle$$
$$+ \bar\lambda_b\langle f\,g\rangle[g\,c]\langle c\,d\rangle + \bar\lambda_b[c|P_{abc}|f\rangle\langle c\,d\rangle + \bar\lambda_b\langle f\,d\rangle s_{ab}. \qquad (E.2)$$

$$D_B[a,b,c,d,e,f,g] =$$

$$- \frac{1}{\langle e\,f\rangle\langle f\,g\rangle\langle a\,b\rangle[c\,d]} G_5[a, g, P_{ab}P_{cd}|e\rangle, P_{ab}|c], P_{efg}|d\rangle; b, e, e, X_{B1a}, X_{B1a}, P_{ab}|d]; d; P_{efg}; P_{ab}]$$

$$+ \frac{\langle b\,c\rangle^2}{\langle a\,b\rangle\langle f\,g\rangle[d\,e]}$$
$$\times H_6[a, g, P_{fg}|e], P_{efg}|d\rangle, P_{fg}P_{de}|c\rangle, P_{defg}P_{de}|f\rangle; b, c, P_{fg}|d], P_{fg}|d], P_{fg}|d], P_{fg}|d]; P_{ab}]$$

$$- \frac{\langle c\,e\rangle^4[a\,f]}{\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle\langle f\,g\rangle t_{cde}}$$
$$\times H_5[a, g, P_{ab}P_{cde}|f\rangle, P_{ab}P_{cde}|e\rangle, P_{fg}P_{cde}|c\rangle; b, a, f, P_{fg}P_{cde}|b\rangle, P_{fg}P_{cde}|b\rangle; P_{ab}]$$

$$- \frac{\langle c\,e\rangle^4[b\,f]}{\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle\langle f\,g\rangle t_{cde}}$$
$$\times H_5[a, g, P_{ab}P_{cde}|f\rangle, P_{ab}P_{cde}|e\rangle, P_{fg}P_{cde}|c\rangle; b, b, f, P_{fg}P_{cde}|b\rangle, P_{fg}P_{cde}|b\rangle; P_{ab}]$$

$$- \frac{\langle c\,e\rangle^4[a\,g]}{\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle\langle f\,g\rangle t_{cde}}$$
$$\times H_5[a, g, P_{ab}P_{cde}|f\rangle, P_{ab}P_{cde}|e\rangle, P_{fg}P_{cde}|c\rangle; b, a, g, P_{fg}P_{cde}|b\rangle, P_{fg}P_{cde}|b\rangle; P_{ab}]$$

$$- \frac{\langle c\,e\rangle^4[b\,g]}{\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle\langle f\,g\rangle t_{cde}}$$
$$\times H_5[a, g, P_{ab}P_{cde}|f\rangle, P_{ab}P_{cde}|e\rangle, P_{fg}P_{cde}|c\rangle; b, b, g, P_{fg}P_{cde}|b\rangle, P_{fg}P_{cde}|b\rangle; P_{ab}]$$

$$+ \frac{\langle c\,e\rangle^4[g\,a]^2\langle a\,b\rangle^2}{\langle c\,d\rangle\langle d\,e\rangle\langle e\,f\rangle\langle a\,b\rangle s_{ab}[g|P_{gab}|c\rangle t_{gab}} H_2[a, P_{ab}P_{gab}|f\rangle; b, P_{ab}|g]; P_{ab}]$$

$$- \frac{[g|P_{abc}|e\rangle^4\langle b\,c\rangle^2}{\langle a\,b\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{abc}|c\rangle[g|P_{abc}|d\rangle t_{abc}t_{def}} H_2[a, P_{abc}P_{de}|f\rangle; b, c; P_{ab}]$$

$$- \frac{[f\,g]^3\langle b\,c\rangle^2}{\langle a\,b\rangle\langle c\,d\rangle[e\,f]t_{efg}[g|P_{ef}|d\rangle} H_2[a, P_{fg}|e]; b, c; P_{ab}], \qquad (E.3)$$

where,

$$|X_{B1a}\rangle = [d\,a]\langle b\,e\rangle|a\rangle + [d|P_{bcd}|e\rangle|b\rangle. \qquad (E.4)$$

$$D_C[a,b,c,d,e,f,g]$$

$$= \frac{1}{[b\,c]\langle d\,e\rangle\langle e\,f\rangle\langle g\,a\rangle} G_5[P_{ga}|b], f, P_{def}|c], P_{ga}P_{bc}|d\rangle, g; e, e, X_{C1a}, X_{C1a}, a, P_{ga}|c]; c; P_{bc}; P_{ga}]$$

$$- \frac{[c\,d]^3\langle b\,a\rangle^2}{\langle e\,f\rangle\langle g\,a\rangle} H_6[f, g, P_{gab}P_{cd}|e\rangle, P_{ef}P_{cd}|b\rangle, P_{def}|d], P_{def}|c]; a, b, e, e, e, e; P_{ga}]$$

$$- \frac{1}{\langle e\,f\rangle\langle b\,c\rangle\langle c\,d\rangle\langle g\,a\rangle t_{bcd}}$$

$$\times G_5[f, g, P_{ef}P_{cd}|b\rangle, P_{ga}P_{bcd}|e\rangle, P_{ga}P_{bcd}|d\rangle; a, e, e, X_{C1c}, X_{C1c}, P_{ga}P_{bcd}|b\rangle; P_{bcd}|b]; P_{ef}; P_{ga}]$$

$$- \frac{\langle b\,e\rangle^4\langle a\,g\rangle^2[g\,f]^2}{\langle b\,c\rangle\langle c\,d\rangle\langle d\,e\rangle\langle g\,a\rangle[f|P_{cde}|b\rangle t_{fga}s_{ga}} \overline{H}_2[a, P_{bcd}|e\rangle; f, g; P_{ga}]$$

$$+ \frac{[f|P_{cd}|e\rangle^4\langle b\,a\rangle^2}{\langle c\,d\rangle\langle d\,e\rangle[f|P_{de}|c][f|P_{ga}|b\rangle t_{gab}t_{cde}\langle g\,a\rangle} H_2[g, P_{gab}P_{cd}|e\rangle; a, b; P_{ga}]$$

$$- \frac{[f\,d]^4\langle b\,a\rangle^2}{\langle b\,c\rangle[d\,e][e\,f][f|P_{de}|c\rangle\langle g\,a\rangle t_{def}} H_2[g, P_{ef}|d]; a, b; P_{ga}],$$

$$\text{(E.5)}$$

where,

$$|X_{C1a}\rangle = -\langle e\,a\rangle([c\,g]|g\rangle + [c\,a]|a\rangle) + [c\,b]\langle b\,e\rangle|a\rangle,$$

$$|X_{C1c}\rangle = \langle e\,b\rangle t_{bcd}|a\rangle + \langle e\,a\rangle(\langle b|P_{cd}|g]|g\rangle + \langle b|P_{cd}|a]|a\rangle). \quad \text{(E.6)}$$

$$D_D[a,b,c,d,e,f,g] =$$

$$\frac{1}{\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle[f\,g]} G_4[b, P_{ab}|g], P_{ab}P_{fg}|e\rangle, P_{cde}|f]; a, c, X_{D1}, X_{D1}, P_{ab}|f]; f; P_{cde}; P_{ab}]$$

$$- \frac{\langle a\,g\rangle^2[f|P_{cde}|c\rangle^3}{\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle\langle e|P_{cd}P_{ab}|g\rangle t_{cde}t_{gab}} H_2[b, P_{cde}|f]; a, g; P_{ab}]$$

$$- \frac{[d|P_{ef}|g\rangle^3\langle a\,g\rangle^2}{\langle a\,b\rangle\langle e\,f\rangle\langle f\,g\rangle[c\,d][c|P_{ab}|g\rangle\langle g|P_{ab}P_{cd}|e\rangle} H_2[b, P_{cd}P_{ef}|g\rangle; a, g; P_{ab}]$$

$$+ \frac{1}{\langle a\,b\rangle\langle e\,f\rangle\langle f\,g\rangle\langle c\,d\rangle t_{efg}}$$

$$\times G_4[b, P_{ab}P_{efg}|e\rangle, P_{ab}P_{efg}|d\rangle, P_{cd}P_{ef}|g\rangle; a, c, P_{ab}P_{ef}|g\rangle, X_{D4}, X_{D4}; P_{ef}|g\rangle; P_{cd}; P_{ab}]$$

$$- \frac{[b|P_{def}|g\rangle^2}{[a\,b]\langle d\,e\rangle\langle e\,f\rangle\langle f\,g\rangle[c|P_{abc}|g\rangle t_{abc}} \overline{H}_3[a, P_{abc}|d\rangle, c; b, P_{abc}|g\rangle, P_{abc}|g\rangle; P_{ab}], \quad \text{(E.7)}$$

where,

$$|X_{D1}\rangle = -|a\rangle[f|P_{ga}|c\rangle + |b\rangle[b\,f]\langle a\,c\rangle,$$

$$|X_{D4}\rangle = |a\rangle[d|P_{ef}|g\rangle\langle c\,d\rangle + |c\rangle[b|P_{ef}|g\rangle\langle a\,b\rangle. \quad \text{(E.8)}$$

$$D_E[a,b,c,d,e,f,g] =$$

$$- \frac{1}{\langle c\,d\rangle\langle e\,f\rangle\langle f\,g\rangle s_{ab}}$$

$$\times \begin{pmatrix} [f|P_{eb}|d\rangle^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,f,b,f;a;P_{efg};P_{ab}] \\ +[g|P_{eb}|d\rangle^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,g,b,g;a;P_{efg};P_{ab}] \\ +([f\,a]\langle b\,d\rangle)^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,a,f,a,f;a;P_{efg};P_{ab}] \\ +([g\,a]\langle b\,d\rangle)^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,a,g,a,g;a;P_{efg};P_{ab}] \\ +2[f|P_{eb}|d\rangle[g|P_{eb}|d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,f,b,g;a;P_{efg};P_{ab}] \\ +2[f|P_{eb}|d\rangle[f\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,f,a,f;a;P_{efg};P_{ab}] \\ +2[f|P_{eb}|d\rangle[g\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,f,a,g;a;P_{efg};P_{ab}] \\ +2[g|P_{eb}|d\rangle[f\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,g,a,f;a;P_{efg};P_{ab}] \\ +2[g|P_{eb}|d\rangle[g\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,b,g,a,g;a;P_{efg};P_{ab}] \\ +2[f\,a]\langle b\,d\rangle[g\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,e,a,f,a,g;a;P_{efg};P_{ab}] \end{pmatrix}$$

$$-\frac{\langle e\,d\rangle}{\langle c\,d\rangle\langle e\,f\rangle\langle f\,g\rangle\langle d\,f\rangle s_{ab}}$$

$$\times \begin{pmatrix} [f|P_{eb}|d\rangle^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,f,b,f;a;P_{fgX_{E3}};P_{ab}] \\ +([g|P_{eb}|d\rangle)^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,g,b,g;a;P_{fgX_{E3}};P_{ab}] \\ +([f\,a]\langle b\,d\rangle)^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,a,f,f,a,f;a;P_{fgX_{E3}};P_{ab}] \\ +([g\,a]\langle b\,d\rangle)^2 & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,a,g,a,g;a;P_{fgX_{E3}};P_{ab}] \\ +2[f|P_{eb}|d\rangle[g|P_{eb}|d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,f,b,g;a;P_{fgX_{E3}};P_{ab}] \\ +2[f|P_{eb}|d\rangle[f\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,f,a,f;a;P_{fgX_{E3}};P_{ab}] \\ +2[f|P_{eb}|d\rangle[g\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,f,a,g;a;P_{fgX_{E3}};P_{ab}] \\ +2[g|P_{eb}|d\rangle[f\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,g,a,f;a;P_{fgX_{E3}};P_{ab}] \\ +2[g|P_{eb}|d\rangle[g\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,b,g,a,g;a;P_{fgX_{E3}};P_{ab}] \\ +2[f\,a]\langle b\,d\rangle[g\,a]\langle b\,d\rangle & G_5[a,g,c,P_{fg}|e],P_{cd}|e];d,f,a,f,a,g;a;P_{fgX_{E3}};P_{ab}] \end{pmatrix}$$

$$+\frac{\langle d\,e\rangle^3}{\langle a\,b\rangle\langle e\,f\rangle\langle f\,g\rangle\langle d\,f\rangle[c|P_{de}|f\rangle}$$

$$\times G_4[g,a,P_{ab}P_{cde}|f\rangle,P_{abc}P_{de}|f\rangle;b,f,f,X_{E1b},X_{E1b};c;P_{fgX_{E3}};P_{ab}]$$

$$+\frac{[c|P_{cde}|b\rangle^2}{\langle f\,g\rangle\langle a\,b\rangle[c\,d][d\,e][c|P_{de}|f\rangle t_{cde}}H_3[a,g,P_{cd}|e];b,P_{de}|c],P_{de}|c];P_{ab}]$$

$$-\frac{\langle d\,e\rangle^4[g\,a]^2}{\langle c\,d\rangle\langle d\,e\rangle\langle e\,f\rangle[a\,b][g|P_{ab}|c\rangle t_{gab}}H_2[a,P_{ab}P_{gab}|f\rangle;b,P_{ab}|g];P_{ab}]$$

$$-\frac{\langle d\,e\rangle^3[g|P_{def}|b\rangle^2}{\langle a\,b\rangle\langle e\,f\rangle[g|P_{def}|c\rangle[g|P_{ef}|d\rangle t_{abc}t_{def}}H_3[a,c,P_{abc}P_{de}|f\rangle;b,P_{def}|g],P_{def}|g];P_{ab}]$$

$$-\frac{[f\,g]^3\langle b\,d\rangle^2}{\langle a\,b\rangle\langle c\,d\rangle[e\,f][g|P_{ef}|d\rangle t_{efg}}H_3[a,c,P_{fg}|e];b,d,d;P_{ab}]\,,\tag{E.9}$$

where,

$$|X_{E1b}\rangle = [c\,g]\langle g\,f\rangle|b\rangle + [c\,a]\langle a\,b\rangle|f\rangle\,,$$
$$\not{k}_{X_{E3}} = \frac{\langle e\,d\rangle}{\langle f\,d\rangle}|f\rangle\langle e|\,. \tag{E.10}$$

$$D_F[a,b,c,d,e,f,g]$$
$$=\frac{1}{\langle a\,b\rangle[c\,d]\langle e\,f\rangle\langle f\,g\rangle}G_4[b,g,P_{efg}|d],P_{ab}P_{cd}|e\rangle;f,f,a,X_{F1a},X_{F1a};c;P_{cd};P_{ab}]$$

$$- \frac{[e\,c]}{\langle f\,g\rangle[d\,e]\langle a\,b\rangle}$$
$$\times H_7[b,g,c,P_{fg}|e\rangle, P_{fg}P_{de}|c\rangle, P_{abc}|d], P_{abc}P_{de}|f\rangle; a,c,f,f,P_{abc}|e], X_{F1b}, X_{F1b}; P_{ab}]$$
$$+ \frac{1}{\langle f\,g\rangle[d\,e]\langle a\,b\rangle}$$
$$\times H_7[b,c,g,P_{fg}P_{de}|c\rangle, P_{abc}P_{de}|f\rangle, P_{abc}|d], P_{fg}|e]; a,f,f,P_{abc}|e], P_{ab}|e], X_{F1b}, X_{F1b}; P_{ab}]$$
$$- \frac{1}{\langle f\,g\rangle\langle a\,b\rangle\langle c\,d\rangle\langle d\,e\rangle t_{cde}}$$
$$\times G_5[b,g,P_{fg}P_{de}|c\rangle, P_{ab}P_{cd}|e\rangle, P_{ab}P_{cde}|f\rangle; a,f,f,P_{ab}P_{cde}|d\rangle, X_{F1c}, X_{F1c}; P_{cde}|d]; P_{fg}; P_{ab}]$$
$$- \frac{\langle d\,f\rangle^4[g\,b]^2\langle b\,a\rangle}{\langle c\,d\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{def}|c\rangle s_{ab}t_{gab}}\overline{H}_2[P_{ab}|b\rangle, P_{cde}|f\rangle; g, P_{ab}|a\rangle; P_{ab}]$$
$$- \frac{\langle d\,f\rangle^4\langle g|P_{abc}|a]^2}{\langle a\,b\rangle\langle d\,e\rangle\langle e\,f\rangle[g|P_{abc}|c\rangle[g|P_{abc}|d\rangle t_{abc}t_{def}}H_3[b,c,P_{abc}P_{de}|f\rangle; P_{abc}|g], P_{abc}|g], a; P_{ab}]$$
$$- \frac{[g\,e]^4\langle d\,a\rangle^2}{\langle a\,b\rangle\langle c\,d\rangle[e\,f][f\,g][g|P_{efg}|d\rangle t_{efg}}H_3[b,c,P_{efg}|e\rangle; d,d,a; P_{ab}]\,, \tag{E.11}$$

where,

$$|X_{F1a}\rangle = |f\rangle[c\,b]\langle b\,a\rangle + |a\rangle[c|P_{efg}|f\rangle\,,$$
$$|X_{F1b}\rangle = |f\rangle[e\,d]\langle d\,a\rangle + \langle f\,a\rangle(|f\rangle[e\,f] + |g\rangle[e\,g])\,, \tag{E.12}$$
$$|X_{F1c}\rangle = |a\rangle\langle f\,g\rangle[g|P_{cde}|d] + |f\rangle\langle a\,b\rangle[b|P_{cde}|d]\,.$$

References

[1] NLO MULTILEG WORKING GROUP collaboration, Z. Bern et al., *The NLO multileg working group: summary report*, arXiv:0803.0494 [SPIRES].

[2] G. Passarino and M.J.G. Veltman, *One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model*, *Nucl. Phys.* **B 160** (1979) 151 [SPIRES];
R.G. Stuart, *Algebraic reduction of one loop Feynman diagrams to scalar integrals*, *Comput. Phys. Commun.* **48** (1988) 367 [SPIRES];
G.J. van Oldenborgh and J.A.M. Vermaseren, *New algorithms for one loop integrals*, *Z. Phys.* **C 46** (1990) 425 [SPIRES].

[3] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop n-point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226] [SPIRES].

[4] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265] [SPIRES].

[5] R.E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, *J. Math. Phys.* **1** (1960) 429 [SPIRES].

[6] Z. Bern, L.J. Dixon and D.A. Kosower, *One-loop amplitudes for e^+e^- to four partons*, *Nucl. Phys.* **B 513** (1998) 3 [hep-ph/9708239] [SPIRES].

[7] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries*, *Phys. Lett.* **B 612** (2005) 75 [hep-th/0502028] [SPIRES].

[8] D. Forde, *Direct extraction of one-loop integral coefficients*, *Phys. Rev.* **D 75** (2007) 125019 [arXiv:0704.1835] [SPIRES].

[9] N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *Analytic structure of three-mass triangle coefficients*, *JHEP* **04** (2008) 038 [arXiv:0709.2086] [SPIRES].

[10] P. Mastrolia, *On triple-cut of scattering amplitudes*, *Phys. Lett.* **B 644** (2007) 272 [hep-th/0611091] [SPIRES].

[11] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103] [SPIRES].

[12] Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 71** (2005) 045006 [hep-th/0410224] [SPIRES].

[13] Z. Bern, L.J. Dixon and D.A. Kosower, *All next-to-maximally helicity-violating one-loop gluon amplitudes in $N = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 72** (2005) 045014 [hep-th/0412210] [SPIRES].

[14] R. Roiban, M. Spradlin and A. Volovich, *Dissolving $N = 4$ loop amplitudes into QCD tree amplitudes*, *Phys. Rev. Lett.* **94** (2005) 102002 [hep-th/0412265] [SPIRES].

[15] S.J. Bidder, D.C. Dunbar and W.B. Perkins, *Supersymmetric Ward identities and NMHV amplitudes involving gluinos*, *JHEP* **08** (2005) 055 [hep-th/0505249] [SPIRES];
K. Risager, S.J. Bidder and W.B. Perkins, *One-loop NMHV amplitudes involving gluinos and scalars in $N = 4$ gauge theory*, *JHEP* **10** (2005) 003 [hep-th/0507170] [SPIRES].

[16] N.E.J. Bjerrum-Bohr, D.C. Dunbar, H. Ita, W.B. Perkins and K. Risager, *The no-triangle hypothesis for $N = 8$ supergravity*, *JHEP* **12** (2006) 072 [hep-th/0610043] [SPIRES];
Z. Bern, N.E.J. Bjerrum-Bohr and D.C. Dunbar, *Inherited twistor-space structure of gravity loop amplitudes*, *JHEP* **05** (2005) 056 [hep-th/0501137] [SPIRES];
Z. Bern, J.J. Carrasco, D. Forde, H. Ita and H. Johansson, *Unexpected cancellations in gravity theories*, *Phys. Rev.* **D 77** (2008) 025010 [arXiv:0707.1035] [SPIRES];
N.E.J. Bjerrum-Bohr and P. Vanhove, *Absence of triangles in maximal supergravity amplitudes*, *JHEP* **10** (2008) 006 [arXiv:0805.3682] [SPIRES].

[17] E.W. Nigel Glover and C. Williams, *One-loop gluonic amplitudes from single unitarity cuts*, *JHEP* **12** (2008) 067 [arXiv:0810.2964] [SPIRES].

[18] Z. Bern, J.S. Rozowsky and B. Yan, *Two-loop four-gluon amplitudes in $N = 4$ super-Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424] [SPIRES].

[19] Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys.* **B 530** (1998) 401 [hep-th/9802162] [SPIRES].

[20] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205] [SPIRES];
Z. Bern et al., *Three-loop superfiniteness of $N = 8$ supergravity*, *Phys. Rev. Lett.* **98** (2007) 161303 [hep-th/0702112] [SPIRES].

[21] D.C. Dunbar, *One-loop six gluon amplitude*, arXiv:0901.1202 [SPIRES].

[22] S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar, *$N = 1$ supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts*, *Phys. Lett.* **B 606** (2005) 189 [hep-th/0410296] [SPIRES].

[23] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev.* **D 72** (2005) 065012 [hep-ph/0503132] [SPIRES].

[24] Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, *JHEP* **11** (2005) 027 [hep-ph/0507019] [SPIRES].

[25] R. Britto, B. Feng and P. Mastrolia, *The cut-constructible part of QCD amplitudes*, *Phys. Rev.* **D 73** (2006) 105004 [hep-ph/0602178] [SPIRES].

[26] W.L. van Neerven, *Dimensional regularization of mass and infrared singularities in two loop on-shell vertex functions*, *Nucl. Phys.* **B 268** (1986) 453 [SPIRES];
Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, *Nucl. Phys.* **B 467** (1996) 479 [hep-ph/9511336] [SPIRES];
Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop self-dual and $N = 4$ super Yang-Mills*, *Phys. Lett.* **B 394** (1997) 105 [hep-th/9611127] [SPIRES];
A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [hep-th/0506068] [SPIRES];
C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191] [SPIRES]; *Unitarity cuts and reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277] [SPIRES];
R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089] [SPIRES];
W.T. Giele, Z. Kunszt and K. Melnikov, *Full one-loop amplitudes from tree amplitudes*, *JHEP* **04** (2008) 049 [arXiv:0801.2237] [SPIRES];
R.K. Ellis, W.T. Giele, Z. Kunszt and K. Melnikov, *Masses, fermions and generalized D-dimensional unitarity*, arXiv:0806.3467 [SPIRES].

[27] C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *Bootstrapping one-loop QCD amplitudes with general helicities*, *Phys. Rev.* **D 74** (2006) 036009 [hep-ph/0604195] [SPIRES]; *All one-loop maximally helicity violating gluonic amplitudes in QCD*, *Phys. Rev.* **D 75** (2007) 016006 [hep-ph/0607014] [SPIRES].

[28] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [hep-th/0412308] [SPIRES];
R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [hep-th/0501052] [SPIRES].

[29] C.F. Berger et al., *An automated implementation of on-shell methods for one-loop amplitudes*, *Phys. Rev.* **D 78** (2008) 036003 [arXiv:0803.4180] [SPIRES].

[30] Z. Xiao, G. Yang and C.-J. Zhu, *The rational parts of one-loop QCD amplitudes III: the six-gluon case*, *Nucl. Phys.* **B 758** (2006) 53 [hep-ph/0607017] [SPIRES].

[31] T. Binoth, J.P. Guillet and G. Heinrich, *Algebraic evaluation of rational polynomials in one-loop amplitudes*, *JHEP* **02** (2007) 013 [hep-ph/0609054] [SPIRES].

[32] S.D. Badger, *Direct extraction of one loop rational terms*, *JHEP* **01** (2009) 049 [arXiv:0806.4600] [SPIRES].

[33] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B 763** (2007) 147 [hep-ph/0609007] [SPIRES];

CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes, *JHEP* **03** (2008) 042 [`arXiv:0711.3596`] [SPIRES].

[34] J.E. Paton and H.-M. Chan, *Generalized Veneziano model with isospin*, *Nucl. Phys.* **B 10** (1969) 516 [SPIRES];
F.A. Berends and W. Giele, *The six gluon process as an example of Weyl-Van Der Waerden spinor calculus*, *Nucl. Phys.* **B 294** (1987) 700 [SPIRES];
M.L. Mangano, *The color structure of gluon emission*, *Nucl. Phys.* **B 309** (1988) 461 [SPIRES].

[35] Z. Bern and D.A. Kosower, *Color decomposition of one loop amplitudes in gauge theories*, *Nucl. Phys.* **B 362** (1991) 389 [SPIRES].

[36] Z. Xu, D.-H. Zhang and L. Chang, *Helicity amplitudes for multiple bremsstrahlung in massless nonabelian gauge theories*, *Nucl. Phys.* **B 291** (1987) 392 [SPIRES].

[37] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [`hep-th/0312171`] [SPIRES].

[38] Z. Kunszt, A. Signer and Z. Trócsányi, *Singular terms of helicity amplitudes at one loop in QCD and the soft limit of the cross-sections of multiparton processes*, *Nucl. Phys.* **B 420** (1994) 550 [`hep-ph/9401294`] [SPIRES].

[39] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *Twistor space structure of the box coefficients of $N = 1$ one-loop amplitudes*, *Phys. Lett.* **B 608** (2005) 151 [`hep-th/0412023`] [SPIRES].

[40] H.J. Lu and C.A. Perez, *Massless one loop scalar three point integral and associated Clausen, Glaisher and L functions*, SLAC-PUB-5809 (1992) [SPIRES];
Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B 412** (1994) 751 [`hep-ph/9306240`] [SPIRES].